Foliated Lie and Courant Algebroids

by

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ABSTRACT. If A is a Lie algebroid over a foliated manifold $(M, \mathcal{F})$, a foliation of A is a Lie subalgebroid B with anchor image $T\mathcal{F}$ and such that A/B is locally equivalent with Lie algebroids over the slice manifolds of $\mathcal{F}$. We give several examples and, for foliated Lie algebroids, we discuss the following subjects: the dual Poisson structure and Vaintrob's super-vector field, cohomology and deformations of the foliation, integration to a Lie groupoid. In the last section, we define a corresponding notion of a foliation of a Courant algebroid A as a bracket-closed, isotropic subbundle B with anchor image $T\mathcal{F}$ and such that $B^{\perp}/B$ is locally equivalent with Courant algebroids over the slice manifolds of $\mathcal{F}$. Examples that motivate the definition are given.

The main categories of interest for Differential Geometry are the C^∞ category and the complex analytic category. However, there also is a significant interest in the category of foliated manifolds. On the other hand, in the last thirty years Lie algebroids became a central theme of differential-geometric research, usually within the framework of the C^∞ category. Recently, a general study of holomorphic Lie algebroids has also been done (see [9, 10]). The aim of the present paper is to start a similar study of Lie algebroids in the C^∞-foliated category.

Essentially, we will say that a Lie subalgebroid B is a foliation of the Lie algebroid A over the foliated manifold $(M, \mathcal{F})$ if the anchor image of B is $T\mathcal{F}$ and A/B is locally equivalent with Lie algebroids over the slice manifolds of $\mathcal{F}$. In Section 1, after giving the precise definitions and first properties, we discuss several examples of foliated Lie algebroids (A, B): classical foliations, transitive foliations that correspond to foliated principal bundles, foliated Dirac structures, etc. In Section 2, we show that the tangent Lie algebroid of a foliated Lie algebroid is a foliated Lie algebroid. In Section 3, we establish the characteristic properties of the dual Poisson structure and of Vaintrob's super-vector field of a foliated Lie algebroid. In Section 4, we define the cohomological spectral sequence of a foliated Lie algebroid and prove a Poincaré lemma in a particular case. Then, we define deformations of a foliation B of A and prove the existence of corresponding, cohomological, infinitesimal deformations. In

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Section 5, we show that, if A is integrable to a Lie groupoid G and B is a foliation of A, G has a foliation $\mathcal{G}$ such that the restriction of the construction of the Lie algebroid of A to the leaves of $\mathcal{G}$ produces the Lie subalgebroid B. Finally, in Section 6, we give a corresponding definition of foliated Courant algebroids. With a notation similar to the above, in the Courant case we will ask B to be a bracket-closed, isotropic subbundle of the Courant algebroid A, with anchor image $T\mathcal{F}$, such that $B^{\perp}/B$ is locally equivalent with Courant algebroids over the slice manifolds of $\mathcal{F}$. A usual foliation is a foliation of the Courant algebroid $TM \oplus T^*M$ in this sense.

In the whole paper, we will use the Einstein summation convention.

1 Definitions and examples

We begin by recalling some basic facts concerning foliations; the reader may find details in [18]. Consider an m-dimensional manifold M endowed with a C^∞-foliation $\mathcal{F}$ with n-dimensional leaves and codimension $k = m - n$. Then, each point has local adapted coordinates (x^a, y^u) $(a, b, ... = 1, ..., k;\ u = 1, ..., n)$, with origin-centered, rectangular range, such that the local equations of $\mathcal{F}$ are $x^a = const.$ and y^u are leaf-wise coordinates. The foliation has the following, associated exact sequence of vector bundles

$$(1.1) \qquad 0 \to T\mathcal{F} \xrightarrow{\iota} TM \xrightarrow{\psi} \nu\mathcal{F} \to 0,$$

where $T\mathcal{F}$ is the tangent bundle of the leaves, which will also be denoted by F, and $\nu\mathcal{F} = TM/F$ is the transversal bundle of the foliation.

A mapping $\varphi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ between two foliated manifolds is a *foliated mapping* if it sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$. The local equations of such a map are of the form

$$(1.2) \qquad x_2^{a_2} = x_2^{a_2}(x_1^{a_1}),\ y_2^{u_2} = y_2^{u_2}(x_1^{a_1}, y_1^{u_1}).$$

The mapping φ is called a *leaf-wise immersion, submersion, local diffeomorphism* if the morphism induced by its differential φ_* between the tangent bundles $T\mathcal{F}_1, T\mathcal{F}_2$ is an injection, surjection, isomorphism, respectively. There also exists a similar notion of a *transversal immersion, submersion, local diffeomorphism* that refers to the transversal bundles $\nu\mathcal{F}_1, \nu\mathcal{F}_2$.

Let $\pi_{P\to M} : P \to (M, \mathcal{F})$ (notice our notation for projections of bundles onto the base manifold; the index of π will be omitted if no confusion is feared) be a principal G-bundle over the foliated manifold $(M, \mathcal{F})$. P is a *foliated bundle* if it is endowed with a foliation $\mathcal{F}^P$ that satisfies the conditions:

(i) $T\mathcal{F}^P$ has a trivial intersection with the tangent spaces of the fibers (i.e., the intersection is equal to 0 and $T\mathcal{F}^P$ is "horizontal"),

(ii) $\mathcal{F}^P$ is invariant by G-right translations,

(iii) the projection π is a foliated leaf-wise submersion.

In fact, because of (ii) π induces a covering map between the leaves of $\mathcal{F}^P$ and the corresponding leaves of $\mathcal{F}$. Equivalently, a foliated principal bundle

may be characterized by an atlas of local trivializations with G-valued, foliated transition functions where G is foliated by points. A foliated principal bundle P may be seen as the glue-up of pullbacks of G-principal bundles on the local transversal manifolds of the leaves of $\mathcal{F}$.

The associated bundles of a foliated principal bundle are called foliated bundles too; they also have an induced foliation with leaves that cover the leaves of $\mathcal{F}$. Accordingly, a vector bundle is foliated if the corresponding principal bundle of frames is foliated. It follows that a foliated structure on a vector bundle $\pi : A \to M$ is a maximal local-trivialization atlas, defined on neighborhoods $\{U\}$, where the local bases (b_i) are related by matrices of $\mathcal{F}$-foliated functions. Then, A has the $\mathcal{F}$-covering foliation $\mathcal{F}^A$ and a cross section s of A is foliated if the corresponding mapping $s : (M,\mathcal{F}) \to (A,\mathcal{F}^A)$ is foliated. Over a trivializing neighborhood U we have $s = \alpha^i b_i$ where α^i are $\mathcal{F}$-foliated functions. Furthermore, we have a sheaf $\Phi_{\mathcal{F}}(A)$ of germs of foliated cross-sections, which is a locally free sheaf-module over the sheaf $\Phi_{\mathcal{F}}$ of germs of foliated functions on M and which spans the sheaf $\Phi(A)$ of germs of cross sections of A over $C^\infty(M)$. Using the fact that $\Phi(A)$ is a locally free sheaf it follows that every subsheaf $\Phi_{\mathcal{F}}(A) \subseteq \Phi(A)$ that is a locally free sheaf-module over $\Phi_{\mathcal{F}}$ and spans $\Phi(A)$ yields a well defined foliated structure on A. Indeed, let a_i be a basis of local cross sections of A. Since $\Phi_{\mathcal{F}}(A)$ spans $\Phi(A)$, we have $a_i = \sum \beta_\alpha^\lambda b_\lambda$ with the germs of $b_\lambda \in \Phi_{\mathcal{F}}(A)$. The total number of elements b_λ is finite and an independent subsystem may be used as a foliated local basis. Two such bases are related by foliated transition functions.

A foliated structure of A may be identified with a family of vector bundles A_{Q_U} over quotient spaces $Q_U = U/_{U\cap\mathcal{F}}$ such that $A|_U = \pi^{-1}_{U\to Q_U}(A_{Q_U})$ ($\{U\}$ is an open covering of M and π^{-1} denotes bundle pullback). Conversely, a family $\{A_{Q_U}\}$ defines a foliated bundle A if $\pi^{-1}_{U\to Q_U}(A_{Q_U}|_{U\cap V}) = \pi^{-1}_{V\to Q_V}(A_{Q_V}|_{U\cap V})$. If $\mathcal{F}$ consists of the fibers of a submersion $M \to Q$, and if these fibers are assumed to be connected and simply connected, then a foliated bundle A over M is the pullback of a projected bundle $A_Q \to Q$ (see Lemma 2.5 and Proposition 2.7 of [18]).

Finally, a *foliated morphism* $\phi : E_1 \to E_2$ between two foliated vector bundles over the same basis $(M,\mathcal{F})$ is a vector bundle morphism that sends foliated cross sections to foliated cross sections.

The most important example of a foliated vector bundle is the transversal bundle $\nu\mathcal{F}$ of the foliation $\mathcal{F}$. Notice that the foliated cross sections $s \in \Gamma_{fol}\nu\mathcal{F}$ act on foliated functions $f \in C^\infty_{fol}(M,\mathcal{F})$ (the index "fol" is a shortcut for "foliated") by $s(f) = Xf \in C^\infty_{fol}(M,\mathcal{F})$, for every X such that $s = [X]_{\mathrm{mod.}\,F}$ ($s(f)$ is well defined since X is a *foliated vector field*, i.e., a foliated cross section $X : (M,\mathcal{F}) \to (TM, T\mathcal{F})$). Furthermore, on the space $\Gamma_{fol}\nu\mathcal{F}$, there exists a well defined Lie algebra bracket

$$[[X_1]_{\mathrm{mod.}\,F}, [X_2]_{\mathrm{mod.}\,F}]_{\nu\mathcal{F}} = [X_1, X_2]_{\mathrm{mod.}\,F} \tag{1.3}$$

induced by the Lie bracket of the corresponding foliated vector fields.

Accordingly, the definition of a Lie algebroid suggests the following definition.

Definition 1.1. An $\mathcal{F}$-*transversal-Lie algebroid* over $(M, \mathcal{F})$ is a foliated bundle E endowed with a Lie algebra bracket $[\, ,\,]_E$ on $\Gamma_{fol}E$ and a foliated, *transversal anchor* morphism $\sharp_E : E \to \nu\mathcal{F}$ such that

1) $\sharp_E[e_1, e_2]_E = [\sharp_E e_1, \sharp_E e_2]_{\nu\mathcal{F}}, \ \forall e_1, e_2 \in \Gamma_{fol}E,$

2) $[e_1, fe_2]_E = f[e_1, e_2]_E + (\sharp_E e_1(f))e_2, \ \forall e_1, e_2 \in \Gamma_{fol}E, f \in C^\infty_{fol}(M).$

The symbols $\sharp$, $[\, ,\,]$ will be used for any transversal-Lie and Lie algebroid while the index that denotes the bundle will be omitted if there is no risk of confusion. Notice also that we may not ask the morphism $\sharp_E$ to be foliated, a priori; this follows from condition 2), which implies $\sharp e_1(f) \in C^\infty_{fol}(M, \mathcal{F})$, $\forall f \in C^\infty_{fol}(M, \mathcal{F})$. The following result is obvious

Proposition 1.1. *If E is a transversal-Lie algebroid over $(M, \mathcal{F})$, $\nu\mathcal{F}$ projects to the tangent bundles of the local quotient spaces Q_U of $\mathcal{F}$ and the local projected bundles $\{E_{Q_U}\}$ are Lie algebroids. In the case of a submersion $M \to Q$ with connected and simply connected fibers, $E \to M$ projects to a Lie algebroid over Q.*

A transversal-Lie algebroid is not a Lie algebroid, and we shall define the notion of a foliated Lie algebroid as follows.

If A is a Lie algebroid over $(M, \mathcal{F})$ and if B is a Lie subalgebroid, a cross section $a \in \Gamma A$ will be called a *B-foliated cross section* if $[b, a]_A \in \Gamma B$, $\forall b \in \Gamma B$. The correctness of this definition follows from the fact that $[b, a]_A \in \Gamma B$ implies $[fb, a]_A = f[b, a]_A - (\sharp_A a(f))b \in \Gamma B$. We shall denote by $\Gamma_B A$ the space of B-foliated cross sections of A. The Jacobi identity shows that the space $\Gamma_B A$ is closed by A-brackets. If $\sharp_A(B) \subseteq T\mathcal{F}$ then, $\forall a \in \Gamma_B A$ and $\forall f \in C^\infty_{fol}(M, \mathcal{F})$, we have $[b, fa]_A \in \Gamma_B A$. Now, we give the following definition

Definition 1.2. If A is a Lie algebroid on $(M, \mathcal{F})$, a *foliation of A over $\mathcal{F}$* is a Lie subalgebroid $B \subseteq A$ that satisfies the following two conditions: 1) $\sharp_A(B) = F$, 2) locally, ΓA is spanned by $\Gamma_B A$ over $C^\infty(M)$ (this is called the *foliated-generation condition*). A pair (A, B) as above will be called a *foliated Lie algebroid*. If $\sharp_A : B \to F$ is a vector bundle isomorphism, (A, B) is a *minimally foliated Lie algebroid*.

Remark 1.1. The condition that B is closed by A-brackets may be replaced by the Frobenius condition: $\forall \alpha \in ann\, B$, the A-exterior differential $d_A\alpha$ belongs to the ideal of A-forms generated by $ann\, B$.

Lemma 1.1. *If B is a foliation of A and C is a complementary subbundle of B in A ($A = B \oplus C$), ΓC has local bases that consist of B-foliated cross sections of A.*

Proof. Let (c_h) be a local basis of ΓC over an open neighborhood $U \subseteq M$. By the foliated-generation condition we have

$$c_h = \gamma_h^u a_{(h)u} = \gamma_h^u pr_C a_{(h)u},$$

where γ_h^u are differentiable functions and $a_{(h)u}$, therefore, $pr_C a_{(h)u}$ too, are local B-foliated cross sections of A. Since the set $\{pr_C a_{(h)u}\}$ is finite, after shrinking the neighborhood U as necessary, there exists a subset of linearly independent, local cross sections $\{pr_C a_v\}$ such that $c_h = \lambda_h^v a_v pr_C a_v$. This subset obviously is the required local basis of ΓC. □

Remark 1.2. The complementary subbundle C may have elements with projection in F since we did not ask $B = \sharp_A^{-1}(F)$. On the other hand, by condition 1) of Definition 1.2, there exist decompositions $B = ker\, \sharp_B \oplus P$ where P is a subbundle of B.

Proposition 1.2. *If B is a foliation of the Lie algebroid A, then the vector bundle $E = A/B$ is a transversal-Lie algebroid on $(M, \mathcal{F})$.*

Proof. Consider a decomposition $A = B \oplus C$ and take B-foliated local bases of C over an open covering $\{U\}$ of M, which exist by Lemma 1.1. Let $\tilde{a}_v = \lambda_v^w a_w$ be a transition between two such bases and take the bracket by $b \in \Gamma B$. We get $(\sharp b)\lambda_u^v = 0$ and, since $\sharp(B) = F$, the functions λ_u^v must be $\mathcal{F}$-foliated. Thus, C has an induced foliated structure, which transfers to $A/B \approx C$. We notice that $[a]_{\mathrm{mod.}\, B} \in \Gamma_{fol}(A/B)$ if and only if $a \in \Gamma_B A$. Furthermore, the bracket of B-foliated cross sections of A, which is B-foliated too, descends to a well defined Lie bracket on $\Gamma_{fol} E$. Finally, the anchor $\sharp_A$ induces an anchor $\sharp_E : E \to \nu\mathcal{F}$ as required by Definition 1.1. The fact that $\sharp_E$ is a foliated morphism may be justified as in the observation that follows Definition 1.1. Alternatively, we can see that, $\forall a \in \Gamma_B A$, $\sharp_A a$ is a foliated vector field on $(M, \mathcal{F})$ as follows: condition 1) of Definition 1.2 shows that a vector field $Y \in \Gamma F$ is of the form $Y = \sharp_A b$, $b \in \Gamma B$, hence,

$$[Y, \sharp_A a]_A = [\sharp_A b, \sharp_A a]_A = \sharp_A [b, a] \in \Gamma F,$$

which is the required property. □

Corollary 1.1. *A foliated Lie algebroid (A, B) over $(M, \mathcal{F})$ produces a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & B & \overset{j}{\to} & A & \overset{\sigma}{\to} & E & \to & 0 \\ & & \sharp_B \downarrow & & \sharp_A \downarrow & & \sharp_E \downarrow & & \\ 0 & \to & T\mathcal{F} & \overset{\iota}{\to} & TM & \overset{\psi}{\to} & \nu\mathcal{F} & \to & 0 \end{array} \tag{1.4}$$

where the lines are exact sequences of vector bundles.

Definition 1.3. If the transversal-Lie algebroid E is foliately equivalent with a quotient A/B where (A, B) is a foliated Lie algebroid, then E may be placed in a diagram (1.4) and A will be called an *extension* of E. An extension of E such that $B \approx F$ will be called a *minimal extension*.

In what follows we give several examples of foliated Lie algebroids.

Example 1.1. For any foliation $\mathcal{F}$ of a manifold M, $T\mathcal{F}$ is a foliation of the Lie algebroid TM.

Example 1.2. If a Lie algebroid A has a regular subalgebroid B, M has the foliation $\mathcal{F}$ such that $F = \sharp(B)$. Then, (A, B) is a foliated Lie algebroid if the foliated-generation condition is satisfied. For instance, if A is regular, we may take $B = A$ and the foliated-generation condition is trivially satisfied. Another obvious example is that of a regular subalgebroid B such that $B \subseteq ker\,\sharp_A$. Then B is a foliation of A over the foliation of M by points. For instance, let J^1A be the first jet Lie algebroid of A defined in [5]. It is well known that one has the exact sequence

$$0 \to Hom(TM, A) \stackrel{\subseteq}{\to} J^1A \stackrel{\pi}{\to} A \to 0, \tag{1.5}$$

where $\pi(j^1_x a) = a(x)$ $(x \in M, a \in \Gamma A)$ is a Lie algebroid morphism, therefore, $ker\,\pi = Hom(TM, A)$ is a regular Lie subalgebroid of J^1A. This subalgebroid is included in $ker\,\sharp_{J^1A}$ because $j^1_x a \in ker\,\pi_x$ implies $\sharp_{J^1A}(j^1_x a) = \sharp_A(a(x)) = 0$ (the equality $\sharp_{J^1A}(j^1_x a) = \sharp_A a(x)$ is a part of the definition given in [5]). Thus, $Hom(TM, A)$ is a foliation of J^1A over the points of M.

Example 1.3. The typical example of a transitive Lie algebroid is $A = TP/G$, where $\pi : P \to M$ is a principal bundle of structure group G over M and the quotient is the space of the orbits of TP by the action of the differential of the right action of G on P. The cross sections of A may be identified with right-invariant vector fields on P, which allows to define the bracket, and the anchor is induced by $\pi_* : TP \to TM$ [15]. Now, assume that M has the foliation $\mathcal{F}$ and P is a foliated principal bundle with the covering foliation $\mathcal{F}^P$ of $\mathcal{F}$. Then, we get a Lie subalgebroid $B = T\mathcal{F}^P/G \subseteq A$ such that the restriction of the anchor of A to B has the image $T\mathcal{F}$. The foliated-generation condition of A with respect to B is satisfied because it is satisfied for TP with respect to $T\mathcal{F}^P$ and $A/B = (TP/T\hat{\mathcal{F}})/G$. Therefore, (A, B) is a foliated Lie algebroid.

Example 1.4. Following [15], if P is the bundle of frames of a vector bundle $\pi : E \to M$, Example 1.3 has the following equivalent form. Let A be the transitive Lie algebroid $\mathcal{D}(E)$ whose cross sections are the linear differential operators $\Gamma E \to \Gamma E$ of order ≤ 1. The bracket is the commutant of the operators and the anchor is defined by the symbol of the operator. In the presence of a foliation $\mathcal{F}$ on M and if we assume E to be foliated, $\mathcal{D}(E)$ has a Lie subalgebroid $B = \mathcal{D}_{\mathcal{F}}(E)$ defined by the operators whose symbol is a vector field tangent to $\mathcal{F}$ ($\mathcal{D}_{\mathcal{F}}(E)$ is commutant closed since $F = T\mathcal{F}$ is closed by Lie brackets), and the anchor sends B onto the tangent bundle F: if $Y \in \Gamma F$, we get an operator D of symbol Y by putting $Ds = 0$ for $s \in \Gamma_{fol}(E)$ and

$$Dt = \sum (Yf_i)s_i, \ \ \forall t = \sum f_i s_i, \ s_i \in \Gamma_{fol}E, f_i \in C^\infty(M) \tag{1.6}$$

(the definition is correct since if $s \in \Gamma_{fol}(E)$ and $f \in C^\infty_{fol}(M, \mathcal{F})$ (1.6) gives $D(fs) = 0$). The foliated-generation condition is satisfied too. Indeed, an operator of symbol X is of the form $D = \nabla_X + \phi$, where ∇ is a covariant derivative and $\phi \in End(E)$, therefore, $\phi \in \mathcal{D}_{\mathcal{F}}(E)$. The operator D is foliated with respect to $\mathcal{D}_{\mathcal{F}}(E)$ if and only if X is foliated with respect to $\mathcal{F}$ and it

is known that the vector fields on M satisfy the foliated-generation condition. Thus, the pair (A, B) considered above is a foliated Lie algebroid.

Example 1.5. Let D be a Dirac structure on $(M, \mathcal{F})$ such that $F \subseteq D$ (D is $\mathcal{F}$-projectable [25]). Then, (D, F) is a foliated Lie algebroid. The fact that the foliated-generation condition is satisfied was proven in [25] (where, also, concrete examples were given). We notice that there are foliated Dirac structures where F is strictly included in $\sharp_D^{-1}(F)$ ($\sharp_D = pr_{TM}$). For instance,

$$D = F \oplus \{(\sharp_P \alpha, \alpha) \,/\, \alpha \in ann\, F\},$$

where P is a foliated bivector field on M such that the Schouten-Nijenhuis bracket $[P, P]$ vanishes on $ann\, F$ ([25], Example 5.1).

Example 1.6. Let D be a regular Dirac structure on M with the (regular) characteristic foliation $\mathcal{E}$, $E = T\mathcal{E} = pr_{TM} D$ and let $\mathcal{F}$ be a subfoliation of $\mathcal{E}$. Put

$$D_{\mathcal{F}} = D \cap (F \oplus T^*M).$$

Notice that the mapping $\psi : D/D_{\mathcal{F}}(x) \to E/F(x)$ given by $[(X, \xi)]_{\mathrm{mod.}\, D_F} \mapsto [X]_{\mathrm{mod.}\, F}$ is an isomorphism of vector spaces $\forall x \in M$. The existence of this isomorphism shows that $dim\, D_{\mathcal{F}}(x) = const$. Hence, $D_{\mathcal{F}}$ is a vector subbundle of D, which obviously is a Lie subalgebroid of D with anchor image F. Moreover, since $\mathcal{F}$ is a subfoliation of $\mathcal{E}$, the vector bundle E/F is $\mathcal{F}$-foliated and so is its isomorphic image $D/D_{\mathcal{F}}$. This shows that the pair $(D, D_{\mathcal{F}})$ satisfies the foliated-generation condition and is a foliated Lie algebroid.

Proposition 1.3. *For any transversal-Lie algebroid E over the foliated manifold $(M, \mathcal{F})$ and any lift $\rho : E \to TM$ of $\sharp_E$ ($\psi \circ \rho = \sharp_E$, $\psi = pr_{\nu\mathcal{F}}$) there exists a canonical minimal extension $(A_0, \sharp_0, [\,,\,]_0)$ with $A_0 = F \oplus E$, $\sharp_0 = Id + \rho$ and*

$$\begin{aligned} &[Y_1, Y_2]_0 = [Y_1, Y_2],\ [Y, e]_0 = [Y, \rho e],\ [e, Y]_0 = [\rho e, Y] \\ &[e_1, e_2]_0 = ([\rho e_1, \rho e_2] - \rho[e_1, e_2]_E) + [e_1, e_2]_E, \end{aligned} \tag{1.7}$$

$\forall Y, Y_1, Y_2 \in \Gamma F$, $\forall e, e_1, e_2 \in \Gamma_{fol} E$, *where the non-indexed brackets are Lie brackets of vector fields. Conversely, for any minimal extension A of E, there exists a lift ρ such that A is isomorphic to a twisted form of the canonical extension of E by F.*

Proof. The qualification "twisted" is similar to that used in the notion of a twisted Dirac structure and its exact meaning will appear at the end of the proof. We shall use the content and notation of diagram (1.4), which includes the mapping $\psi = pr_{\nu\mathcal{F}}$. Notice that a lift ρ has the property that ρe is a foliated vector field $\forall e \in \Gamma_{fol} E$. If we extend (1.7) by

$$[Y, fe]_0 = f[Y, \rho e] + (Yf)e,\ [e_1, fe_2]_0 = f[e_1, e_2]_0 + \rho e_1(f) e_2,$$

$\forall f \in C^\infty(M)$, we get a skew-symmetric bracket on ΓA_0. It is easy to see that the extension is compatible with (1.7) for $f \in C^\infty_{fol}(M, \mathcal{F})$. Also, straightforward calculations show that the axioms of a Lie algebroid, including the Jacobi

identity, are satisfied on arguments $Y \in \Gamma F, e \in \Gamma_{fol}E$. Since these types of cross sections locally generate ΓA_0, this proves the existence of the canonical extension. Notice that, for $E = \nu\mathcal{F}$, ρ is a splitting of the lower line of (1.4) and, if we identify $F \oplus \approx F \oplus im\,\rho = TM$, the previous construction yields the standard Lie algebroid structure of TM.

For the converse, we start with diagram (1.4), which yields $A = \phi(F) \oplus \tau(E)$ where $\phi : F \approx B, \phi^{-1} = \sharp_B$ (ϕ exists because the extension is minimal) and $\tau : E \to A$ is a splitting of the upper exact sequence of the diagram ($\sigma \circ \tau = Id$). Then, we consider the lift $\rho = \sharp_A \circ \tau$, which implies $\sharp_A = \phi^{-1} + \rho \circ \sigma$. As for the brackets, we must have

$$\phi^{-1}[\phi Y_1, \phi Y_2]_A = \sharp_A[\phi Y_1, \phi Y_2]_A = [\sharp_A \phi Y_1, \sharp_A \phi Y_2] = [Y_1, Y_2],$$

i.e., $[\phi Y_1, \phi Y_2]_A = \phi[Y_1, Y_2]$. Then, since $\sigma[\phi Y, \tau e]_A = [\sigma\phi Y, \sigma\tau e]_E = 0$, $[\phi Y, \tau e]_A \in \Gamma B$ and

$$\phi^{-1}[\phi Y, \tau e]_A = \sharp_A[\phi Y, \tau e]_A = [Y, \sharp_A \tau e] \Leftrightarrow [\phi Y, \tau e]_A = \phi[Y, \rho e], \ \forall e \in \Gamma_{fol}E.$$

Finally, for $e_1, e_2 \in \Gamma_{fol}E$, we have

$$\sharp_A[\tau e_1, \tau e_2]_A - \rho[e_1, e_2]_E = [\rho e_1, \rho e_2] - \rho[e_1, e_2]_E \in \Gamma F$$

because, if we apply ψ to the right hand side of the equality, we get

$$\psi[\rho e_1, \rho e_2] - \sharp_E[e_1, e_2]_E = [\psi\rho e_1, \psi\rho e_2]_{\nu\mathcal{F}} - \sharp_E[e_1, e_2]_E$$

$$= [\sharp_E e_1, \sharp_E e_2]_{\nu\mathcal{F}} - \sharp_E[e_1, e_2]_E = 0$$

by the axioms of a transversal-Lie algebroid. Now, we notice that, if $a \in A$, $\sharp_A a \in \Gamma F$ if and only if $a = \phi Y + \tau e$ where $e \in ker\,\sharp_E$; indeed, by the commutativity of diagram (1.4), $\psi\sharp_A\tau(e) = 0$ is equivalent to $\sharp_E e = 0$. This fact and the previous observation justify the formula

$$[\tau e_1, \tau e_2]_A = ([\tau e_1, \tau e_2]_A - \tau[e_1, e_2]_E) + \tau[e_1, e_2]_E \tag{1.8}$$

$$= [\phi([\rho e_1, \rho e_2] - \rho[e_1, e_2]_E) + \tau(\lambda(e_1, e_2))] + \tau[e_1, e_2]_E,$$

where $\lambda : \wedge^2 E \to E$ is a 2-form with values in the kernel of $\sharp_E$. Therefore, A is isomorphic by $\phi + \tau$ to the canonical extension of E by F twisted by the addition of the form λ. The latter must also be asked to satisfy conditions that ensure the Jacobi identity. □

Remark 1.3. In [3] the authors define a reduction of Lie algebroids that, essentially, is a projection along the fibers of a surjective submersion $\pi : M \to M'$. Namely, $A \to M$ reduces to $A' \to M'$ if there exists a Lie algebroid morphism [15] $(\Pi : A \to A', \pi : M \to M')$. One can see that, if B is a foliation of A over the foliation of M by the fibers of π and if the fibers are connected and simply connected, then A reduces to A' defined by $A/B = \pi^{-1}A'$.

2 The tangent Lie algebroid

In this section we will show that the tangent Lie algebroid of a foliated Lie algebroid (A, B) over a foliated manifold $(M, \mathcal{F})$ has a natural structure of a foliated Lie algebroid over $(TM, T\mathcal{F})$. For this purpose, we formulate the definition of the tangent Lie algebroid of a Lie algebroid [16] in a suitable (but, not new) form.

Let $\pi : A \to M$ be an arbitrary vector bundle. Take a neighborhood $U \subseteq M$ with local coordinates $(x^i)_{i=1}^m$, with the local basis $(a_\alpha)_{\alpha=1}^r$ of ΓA ($r = rankA$) and the corresponding fiber coordinates (ξ^α). On the intersection of two such neighborhoods $U \cap \tilde{U}$, these coordinates have transition functions of the following local form

$$\tilde{x}^i = \tilde{x}^i(x^j),\ \tilde{\xi}^\alpha = \Theta^\alpha_\beta(x^j)\xi^\beta \ \ (\tilde{a}_\alpha = \Psi^\gamma_\alpha a_\gamma,\ \Psi^\gamma_\alpha \Theta^\beta_\gamma = \delta^\beta_\alpha). \tag{2.1}$$

Correspondingly, one has natural coordinates $(x^i, \dot{x}^i)$ on $T_U M$ and $(x^i, \xi^\alpha, \dot{x}^i, \dot{\xi}^\alpha)$ on $T_{\pi^{-1}U}A$, with the following transition functions:

$$\tilde{x}^i = \tilde{x}^i(x^j),\ \tilde{\xi}^\alpha = \Theta^\alpha_\beta(x^j)\xi^\beta,\ \dot{\tilde{x}}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\dot{x}^j,\ \dot{\tilde{\xi}}^\alpha = \frac{\partial \Theta^\alpha_\beta}{\partial x^j}\xi^\beta \dot{x}^j + \Theta^\alpha_\beta \dot{\xi}^\beta. \tag{2.2}$$

Formulas (2.2) show the existence of the double vector bundle

$$\begin{array}{ccc} TA & \stackrel{\pi_*}{\rightarrow} & TM \\ \downarrow & & \downarrow \\ A & \stackrel{\pi}{\rightarrow} & M \end{array} \tag{2.3}$$

where TA is a vector bundle over A with base coordinates (x^i, ξ^α) and fiber coordinates $(\dot{x}^i, \dot{\xi}^\alpha)$ and TA is a vector bundle over TM with base coordinates $(x^i, \dot{x}^i)$ and fiber coordinates $(\xi^\alpha, \dot{\xi}^\alpha)$. (Notice that the cross sections of π_* are not vector fields on A!)

If $A = TM$, the indices $\alpha, \beta, ...$ may be replaced by $i, j, ...$ and $\Theta^i_j = \partial\tilde{x}^i/\partial x^j$, which shows the existence of the flip diffeomorphism $\phi : TTM \to TTM$ defined by the local coordinate equations

$$\phi(x^i, \xi^i, \dot{x}^i, \dot{\xi}^i) = (x^i, \dot{x}^i, \xi^i, \dot{\xi}^i). \tag{2.4}$$

On the fibration $TA \to A$ the fiber coordinates $(\dot{x}^i, \dot{\xi}^\alpha)$ are produced by the local bases of cross sections $(\partial/\partial x^i, \partial/\partial\xi^\alpha)$. On the fibration $\pi_* : TA \to TM$ the fiber coordinates $(\xi^\alpha, \dot{\xi}^\alpha)$ are produced by local bases $(c_\alpha, \partial/\partial\xi^\alpha)$, where the vector c_α is the image of a_α under the natural identification of the tangent space of the fibers of A with the fibers themselves, which are vector spaces. (The previous assertion is justified by checking the invariance of the expression $\xi^\alpha c_\alpha + \dot{\xi}^\alpha(\partial/\partial\xi^\alpha)$ by the coordinate transformations (2.2).) It follows that for $z \in TA$ the intersection space of the fibers of the two vector bundle structures of TA is

$$\mathcal{P}_z = \pi^{-1}_{TA\to A}(\pi_{TA\to A}(z)) \cap (\pi_{A\to M})^{-1}_*((\pi_{A\to M})_*(z)) = span\left\{\frac{\partial}{\partial\xi^\alpha}\right\}. \tag{2.5}$$

Therefore, $P = \cup_{z\in TA}\mathcal{P}_z$ is a vector subbundle of TA, which we call the *bi-vertical subbundle*.

Furthermore, like for the tangent bundle $A = TM$, there are two lifting processes of cross sections of $\pi : A \to M$ to cross sections of $\pi_* : TA \to TM$.

One is the *complete lift* $a \mapsto a^C = a_*$, which sends the cross section a to the differential of the mapping $a : M \to A$. The expression of a_* by means of the local coordinates $(x^i, \dot{x}^i)$ on TM and $(x^i, \dot{x}^i, \xi^\alpha, \dot{\xi}^\alpha)$ on TA is

$$(x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i, \xi^\alpha(x^i), \dot{x}^i \frac{\partial \xi^\alpha}{\partial x^i}),$$

where $a = \xi^\alpha(x^i)a_\alpha$, whence we get

$$a^C = \xi^\alpha c_\alpha + \dot{x}^i \frac{\partial \xi^\alpha}{\partial x^i}\frac{\partial}{\partial \xi^\alpha}. \tag{2.6}$$

The second lift, called the *vertical lift* and denoted by an upper index V, is the isomorphism between the pullback $\pi^{-1}_{TA\to A\to M}A$ of the vector bundle A to TM and the bi-vertical subbundle P defined by

$$a = \eta^\alpha a_\alpha \mapsto a^V = \eta^\alpha \frac{\partial}{\partial \xi^\alpha}. \tag{2.7}$$

(Formulas (2.1) show that the bases (a_α) and $(\partial/\partial\xi^\alpha)$ have the same transition functions.)

Notice the following interpretation of the local basis $(c_\alpha, \partial/\partial\xi^\alpha)$ of TA over TM

$$a^C_\alpha = c_\alpha,\ a^V_\alpha = \frac{\partial}{\partial \xi^\alpha}. \tag{2.8}$$

Notice also the following properties

$$(fa)^C = fa^C + f^C a^V,\ (fa)^V = fa^V \qquad (f \in C^\infty(M),\ f^C = \dot{x}^i \frac{\partial f}{\partial x^i}). \tag{2.9}$$

Now, assume that $(A, \sharp_A, [\,,\,]_A)$ is a Lie algebroid. Define an anchor $\sharp_{TA} : TA \to TTM$ by putting

$$\sharp_{TA} c_\alpha = (\sharp_A a_\alpha)^C,\ \sharp_{TA}\frac{\partial}{\partial \xi^\alpha} = (\sharp_A a_\alpha)^V, \tag{2.10}$$

where in the right hand side the lifts are those of the Lie algebroid TM. A simple calculation shows that $\sharp_{TA} = \phi \circ (\sharp_A)_*$ with ϕ defined by (2.4).

Furthermore, define a bracket of cross sections of π_* by putting

$$[a^V_\alpha, a^V_\beta]_{TA} = 0,\ [a^C_\alpha, a^V_\beta]_{TA} = [a_\alpha, a_\beta]^V_A,\ [a^C_\alpha, a^C_\beta]_{TA} = [a_\alpha, a_\beta]^C_A, \tag{2.11}$$

which yields brackets of the elements of the basis of cross sections of $TA \to TM$, and by extending (2.11) to arbitrary cross sections via the axioms of a Lie

algebroid. One can check that the results of (2.11) hold for the lifts of arbitrary cross sections of A and this justifies the independence of the bracket of the choice of the basis a_α. Then, the axioms of a Lie algebroid hold for the basic cross sections $(c_\alpha, \partial/\partial\xi^\alpha)$, whence, the axioms also hold for any cross sections.

Thus, we have obtained a well defined structure of a Lie algebroid on the bundle $\pi_* : TA \to TM$. This Lie algebroid is called the *tangent Lie algebroid* of A [16].

Now, we prove the announced result

Proposition 2.1. *If the subalgebroid $B \subseteq A$ is a foliation of A over $(M, \mathcal{F})$, then the tangent Lie algebroid TB is a foliation of the Lie algebroid TA over $(TM, T\mathcal{F})$.*

Proof. We may use adapted local coordinates (x^a, y^u) on M, such that $\mathcal{F}$ has the local equations $x^a = 0$, and local bases $(a_h) \equiv (b_\alpha, q_\kappa)$ of ΓA such that (b_α) is a local basis of ΓB and (q_κ) are B-foliated cross sections of A. If we use formulas (2.8), (2.10), (2.11), we see that the complete and vertical lifts of cross sections of $B \subseteq A$ to TA and to TB coincide, and that the bracket and anchor of TB are the restrictions of the bracket and anchor of TA to $TB \subseteq TA$. Therefore, TB is a Lie subalgebroid of TA over TM.

Furthermore, on TM we have local coordinates $(x^a, y^u, \dot{x}^a, \dot{y}^u)$ and (2.2) shows that $x^a = const., \dot{x}^a = const.$ are the equations of a foliation $T\mathcal{F}$ of the manifold TM that consists of the tangent vectors of the leaves of $\mathcal{F}$. Since $\sharp_{TB} = \phi \circ (\sharp_B)_*$ where ϕ is the flip diffeomorphism and $im\,\sharp_B = T\mathcal{F}$, we get $im\,\sharp_{TB} = TT\mathcal{F}$.

Finally, on one hand, (b^C_α, b^V_α) is a local basis of cross sections of TB and, on the other hand, using (2.11), it is easy to check that (q^C_κ, q^V_κ) are TB-foliated, local cross sections of $TA \to TM$. Therefore, since $(b^C_\alpha, q^C_\kappa, b^V_\alpha, q^V_\kappa)$ is a local basis of cross sections of $TA \to TM$, the pair (TA, TB) satisfies the foliated-generation condition. $\square$

3 The dual Poisson structure

It is well known that a Lie algebroid structure on the vector bundle $\pi : E \to M$ is equivalent with a specific Poisson structure on the total space of the dual bundle E^*, called the *dual Poisson structure*, defined by the following brackets of basic and fiber-linear functions:

$$(3.1) \quad \{f_1 \circ \pi, f_2 \circ \pi\} = 0,\ \{f \circ \pi, l_s\} = -((\sharp_E s)f) \circ \pi,\ \{l_{s_1}, l_{s_2}\} = l_{[s_1,s_2]_E},$$

where $f, f_1, f_2 \in C^\infty(M)$, $s, s_1, s_2 \in \Gamma E$ and l_s is the evaluation of the fiber of E^* on s. If x^a are local coordinates on M and η_h are the fiber-coordinates on E^* with respect to the dual of a local basis (e_h) of cross sections of E, then the corresponding Poisson bivector field is

$$(3.2) \qquad \Pi = \frac{1}{2}\alpha^l_{hk}\eta_l \frac{\partial}{\partial\eta_h} \wedge \frac{\partial}{\partial\eta_k} + \alpha^a_h \frac{\partial}{\partial\eta_h} \wedge \frac{\partial}{\partial x^a},$$

where the coefficients α are defined by the expressions

$$\sharp_E(e_h) = \alpha_h^a \frac{\partial}{\partial x^a},\ [e_h, e_k]_E = \alpha_{hk}^l e_l. \tag{3.3}$$

Conversely, formulas (3.2), (3.3) produce an anchor $\sharp_E$ and a bracket $[\,,\,]_E$ and the Poisson condition $[\Pi, \Pi] = 0$ implies the Lie algebroid axioms.

A Poisson structure of the form (3.2) will be called a fiber-linear structure, although this name is not totally appropriate since it does not describe the form of the second term of (3.2). The characteristic property of a bivector field of the form (3.2) is that the Poisson bracket of two fiber-polynomials of degrees h, k is a fiber-polynomial of degree $h + k - 1$.

In this section we establish properties of the dual Poisson structure of a foliated Lie algebroid.

Proposition 3.1. *Let E be a foliated vector bundle over $(M, \mathcal{F})$ with the dual bundle E^*, which has the covering foliation $\mathcal{F}^{E^*}$ of $\mathcal{F}$. A transversal-Lie algebroid structure on E is equivalent with a fiber-linear Poisson algebra structure on the space of $\mathcal{F}^{E^*}$-foliated functions on E^*.*

Proof. Let (x^a, y^u) be $\mathcal{F}$-adapted local coordinates on M and (e_h) be a local foliated basis of E. Then, the space of foliated functions $C^\infty_{fol}(E^*, \mathcal{F}^{E^*})$ is locally spanned by (x^a, η_h) and we get the Poisson algebra structure required by the proposition using formulas (3.1) for $f, f_1, f_2 \in C^\infty_{fol}(M, \mathcal{F}),\ s, s_1, s_2 \in \Gamma_{fol}E$. Conversely, the Poisson structure (3.1) on $C^\infty_{fol}(E^*, \mathcal{F}^{E^*})$ produces a transversal-Lie algebroid on E in the same way as in the case of a Lie algebroid. $\square$

Proposition 3.2. *A foliated Lie algebroid (A, B) over $(M, \mathcal{F})$ is equivalent with a couple (E^*, Λ), where E^* is a vector subbundle of A^* endowed with a foliated structure and the corresponding $\mathcal{F}$-covering foliation $\mathcal{F}^{E^*}$, and Λ is a fiber-linear Poisson structure on the manifold A^* with the following properties:*

1) $\sharp_\Lambda(annTE^*) = T\mathcal{F}^{E^*}$,

2) $\Lambda|_{E^*}$ induces a well defined Poisson algebra structure on $C^\infty_{fol}(E^*, \mathcal{F}^{E^*})$,

3) the total bundle manifold of A^*/E^* has a Poisson structure P such that the projection $(A^*, \Lambda) \to (A^*/E^*, P)$ is a Poisson mapping.

Proof. For the foliated Lie algebroid (A, B), we take $E^* = ann\, B$, which is foliated since its dual bundle is $E = A/B$ and it is foliated. Furthermore, we take the dual Poisson structure Λ of the Lie algebroid A.

In order to write down the Poisson bivector field Λ we define convenient local coordinates on the manifold A^* as follows. We take $\mathcal{F}$-adapted local coordinates (x^a, y^u) on M, we choose a splitting

$$A = B \oplus C, \tag{3.4}$$

and we take a local basis of ΓA that consists of the basis b_h of ΓB and the basis a_q of ΓC where a_q are foliated cross sections (see Lemma 1.1). For A^*, we have the dual bases (b^{*h}, a^{*q}) and corresponding fiber coordinates (η_h, ζ_q).

Since (A,B) is a foliated Lie algebroid, the anchor and bracket have the following local expression

$$\begin{array}{ll}(3.5) & \sharp_A(b_h)=\beta^u_h\frac{\partial}{\partial y^u},\ \sharp_A(a_q)=\alpha^a_q\frac{\partial}{\partial x^a}+\alpha^u_q\frac{\partial}{\partial y^u}\ (rank(\beta^u_h)=dim\,\mathcal{F}),\\ & [b_h,b_k]_A=\sum\beta^l_{hk}b_l,\ [b_h,a_q]_A=\gamma^l_{hq}b_l,\ [a_p,a_q]_A=\alpha^l_{pq}b_l+\alpha^s_{pq}a_s,\end{array}$$

where α^a_q,α^s_{pq} are foliated functions, i.e., locally, these are functions on the coordinates x^a.

Then, the general formula (3.2) implies

$$(3.6)\quad \Lambda=\frac{1}{2}\beta^l_{hk}\eta_l\frac{\partial}{\partial\eta_h}\wedge\frac{\partial}{\partial\eta_k}+\gamma^l_{hq}\eta_l\frac{\partial}{\partial\eta_h}\wedge\frac{\partial}{\partial\zeta_q}+\frac{1}{2}(\alpha^l_{pq}\eta_l+\alpha^s_{pq}\zeta_s)\frac{\partial}{\partial\zeta_p}\wedge\frac{\partial}{\partial\zeta_q}$$
$$+\alpha^a_q\frac{\partial}{\partial\zeta_q}\wedge\frac{\partial}{\partial x^a}+\beta^u_h\frac{\partial}{\partial\eta_h}\wedge\frac{\partial}{\partial y^u}+\alpha^u_q\frac{\partial}{\partial\zeta_q}\wedge\frac{\partial}{\partial y^u}.$$

The total space of the subbundle $E^*\subseteq A^*$ has the local equations $\eta_h=0$, whence

$$\sharp_\Lambda(ann\,TE^*)=span\{i(d\eta_h)\Lambda|_{\eta^h=0}\}=span\{\beta^u_h\frac{\partial}{\partial y^u}\}=T\mathcal{F}^{E^*},$$

which is property 1).

The space $C^\infty_{fol}(E^*,\mathcal{F}^{E^*})$ is locally generated by x^a,ζ_q $(mod.\,\eta_h=0)$ and a function of local expression $f(x^a,\zeta_q)$ extends to a function of local expression $\tilde f(x^a,\zeta_q,\eta_h)$ in a neighborhood of E^* in A^*. Formula (3.6) restricted to $\eta_h=0$ shows that, for $f_1,f_2\in C^\infty_{fol}(E^*,\mathcal{F}^{E^*})$, $\Lambda|_{E^*}(d\tilde f_1,d\tilde f_2)$ depends only on f_1,f_2 and that

$$\{x^a,x^b\}_{\Lambda|_{E^*}}=0,\ \{\zeta_q,x^a\}_{\Lambda|_{E^*}}=\alpha^a_q(x^b),\ \{\zeta_p,\zeta_q\}_{\Lambda|_{E^*}}=\alpha^s_{pq}(x^b)\zeta_s.$$

Therefore, Λ satisfies property 2).

Since $A^*/E^*\approx B^*$, we may use as local coordinates on the manifold A^*/E^* the coordinates (x^a,y^u,η_h). Then,

$$(3.7)\qquad P=\frac{1}{2}\beta^l_{hk}\eta_l\frac{\partial}{\partial\eta_h}\wedge\frac{\partial}{\partial\eta_k}+\beta^u_h\frac{\partial}{\partial\eta_h}\wedge\frac{\partial}{\partial y^u}$$

is a bivector field on A^*/E^*, which is Poisson because it has the same expression as the dual Poisson structure of the Lie algebroid B. Property 3) is obviously satisfied.

Conversely, let A be a vector bundle over $(M,\mathcal{F})$ such that there exists a foliated subbundle E^* of the dual bundle A^* and a fiber-linear Poisson bivector field Λ of A^* with the properties 1), 2), 3). Define the subbundle $B=annE^*\subseteq A$ and take a decomposition $A=B\oplus C$ and local bases of cross sections $b_h\in B,a_q\in C$. Moreover, $C\approx A/B\approx E=E^{**}$ is an $\mathcal{F}$-foliated bundle and a_q may be assumed to be $\mathcal{F}$-foliated cross sections. The corresponding local coordinates

$(x^a, y^u, \eta_h, \zeta_q)$ on A^* are similar to those used in (3.6) and, a priori, Λ is of the form

$$\Lambda = \frac{1}{2}(\beta^l_{hk}\eta_l + \beta^s_{hk}\zeta_s)\frac{\partial}{\partial\eta_h} \wedge \frac{\partial}{\partial\eta_k} + (\gamma^l_{hq}\eta_l + \gamma^s_{hq}\zeta_s)\frac{\partial}{\partial\eta_h} \wedge \frac{\partial}{\partial\zeta_q}$$

$$+\frac{1}{2}(\alpha^l_{pq}\eta_l + \alpha^s_{pq}\zeta_s)\frac{\partial}{\partial\zeta_p} \wedge \frac{\partial}{\partial\zeta_q} + \alpha^a_q\frac{\partial}{\partial\zeta_q} \wedge \frac{\partial}{\partial x^a} + \beta^u_h\frac{\partial}{\partial\eta_h} \wedge \frac{\partial}{\partial y^u}$$

$$+\alpha^u_q\frac{\partial}{\partial\zeta_q} \wedge \frac{\partial}{\partial y^u} + \lambda^a_h\frac{\partial}{\partial\eta_h} \wedge \frac{\partial}{\partial x^a}.$$

If we put $\eta_h = 0$ and compute $i(d\eta_h)\Lambda$ we see that property 1) implies

$$\beta^s_{hk} = 0,\ \gamma^s_{hq} = 0,\ \lambda^a_h = 0,\ rank(\beta^u_h) = dim\,\mathcal{F}.$$

Furthermore, property 2) implies local expressions $\alpha^a_q = \alpha^a_q(x^b)$, $\alpha^s_{pq} = \alpha^s_{pq}(x^b)$. Finally, the projection $(A^* \to A^*/E^*)$-related bivector field P of property 3) is uniquely defined by Λ; it must be given by (3.7) and it satisfies the Poisson condition. Therefore, Λ reduces to the form (3.6) and the corresponding Lie algebroid structure of A is given by formulas (3.5). Accordingly, the pair (A, B) is a foliated Lie algebroid. (The foliated-generation condition is implied by the fact that $[b_h, a_q] \in B$.) $\square$

A. Vaintrob [21] gave an interpretation of the dual Poisson structure (3.2) as an odd, homological super-vector field, which led to important results on Lie algebroids seen as homological super-vector fields. We shall indicate the properties of this field in the foliated case.

The *parity-changed vector bundle* of A^* (in our case A^* is the dual of a Lie algebroid, but, the definition applies to any vector bundle) is a supermanifold ΠA^*, with local even coordinates x^i and local odd coordinates $\bar\eta^\alpha$ associated to local, A^*-trivializing, coordinate neighborhoods U on M, with the following local transition functions:

$$\tilde{x}^i = \tilde{x}^i(x^j),\ \tilde{\bar\eta}^\alpha = b^\alpha_\beta\bar\eta^\beta,$$

where the first equations are the same as the transition functions of the local coordinates on M and the matrix (b^α_β) is defined by a change of local bases of cross sections of A, $a_\beta = b^\alpha_\beta\tilde{a}_\alpha$. The supertangent bundle $T\Pi A^*$ has the local bases $(\partial/\partial x^i,\ \partial/\partial\bar\eta^\alpha)$ with the local transition functions

$$\frac{\partial}{\partial\tilde{x}^i} = \frac{\partial x^j}{\partial\tilde{x}^i}\frac{\partial}{\partial x^j},\ \frac{\partial}{\partial\bar\eta^\beta} = b^\alpha_\beta\frac{\partial}{\partial\tilde{\bar\eta}^\alpha}.$$

These transition functions show that the correspondence

$$x^i \mapsto x^i,\ \frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial x^i},\ \eta_\alpha \mapsto \frac{\partial}{\partial\bar\eta^\alpha},\ \frac{\partial}{\partial\eta_\alpha} \mapsto \bar\eta^\alpha$$

produces a well defined mapping of the fiber-wise polynomial multivector fields on the manifold A^* to super-multivector fields on ΠA^*. In particular, the bivector field Π defined by (3.2) is send to Vaintrob's super-vector field

$$V = \frac{1}{2}\alpha^l_{hk}\bar{\eta}^h\bar{\eta}^k\frac{\partial}{\partial\bar{\eta}^l} + \alpha^a_h\bar{\eta}^h\frac{\partial}{\partial x^a} \tag{3.8}$$

and the Poisson condition $[\Pi,\Pi] = 0$ becomes the homology condition $[V,V] = 0$. To put terminology in agreement with the theory of Lie algebroids, the super-vector field V will be called *transitive* if the *M-rank* ρ of V defined by $\rho = rank(\alpha^a_h)$ is equal to the dimension of M.

Furthermore, if M is endowed with the foliation $\mathcal{F}$ and A has a subbundle B, we have coordinates (x^a, y^u) and bases (b_h, a_q) like in the proof of Proposition 3.2, which yield coordinates $(x^a, y^u, \bar{\eta}^h, \bar{\zeta}^q)$ of ΠA^* with transition functions of the form

$$\tilde{x}^a = \tilde{x}^a(x^b),\ \tilde{y}^u = \tilde{y}^u(x^a, y^v),\ \tilde{\bar{\eta}}^h = b^h_k\bar{\eta}^k + b^h_q\bar{\zeta}^q,\ \tilde{\bar{\zeta}}^q = b^q_p\bar{\zeta}^p. \tag{3.9}$$

where the various matrices b come from a change of the local basis. The transition functions (3.9) show that ΠA^* is endowed with the superfoliation

$$\bar{\mathcal{F}} = span\left\{\frac{\partial}{\partial y^u}, \frac{\partial}{\partial\bar{\eta}^h}\right\}.$$

Then, if we consider the vector bundle $E = A/B$, we have $E^* = ann\, B \subseteq A^*$ and $\Pi(A^*/E^*)$ may be seen as the submanifold of ΠA^* that has the local equations $\bar{\zeta}^q = 0$. On the other hand, ΠE^* is a supermanifold with the local coordinates $(x^a, y^u, \bar{\zeta}^q)$ and there exists a natural projection $\psi : \Pi A^* \to \Pi E^*$ locally defined by

$$(x^a, y^u, \bar{\zeta}^q, \bar{\eta}^h) \mapsto (x^a, y^u, \bar{\zeta}^q).$$

In Vaintrob's terminology a super-vector field on ΠA^* that corresponds to a bivector field on A^* is said to be of degree 1. In our case, a super-vector field of degree 1 has the form

$$\begin{aligned} W = {} & \frac{1}{2}\beta^l_{hk}\bar{\eta}^h\bar{\eta}^k\frac{\partial}{\partial\bar{\eta}^l} + \frac{1}{2}\beta^s_{hk}\bar{\eta}^h\bar{\eta}^k\frac{\partial}{\partial\bar{\zeta}^s} + \gamma^l_{hq}\bar{\eta}^h\bar{\zeta}^q\frac{\partial}{\partial\bar{\eta}^l} \\ & + \gamma^s_{hq}\bar{\eta}^h\bar{\zeta}^q\frac{\partial}{\partial\bar{\zeta}^s} + \frac{1}{2}\alpha^l_{pq}\bar{\zeta}^p\bar{\zeta}^q\frac{\partial}{\partial\bar{\eta}^l} + \frac{1}{2}\alpha^s_{pq}\bar{\zeta}^p\bar{\zeta}^q\frac{\partial}{\partial\bar{\zeta}^s} \\ & + \alpha^a_q\bar{\zeta}^q\frac{\partial}{\partial x^a} + \beta^u_h\bar{\eta}^h\frac{\partial}{\partial y^u} + \alpha^u_q\bar{\zeta}^q\frac{\partial}{\partial y^u} + \lambda^a_h\bar{\eta}^h\frac{\partial}{\partial x^a}. \end{aligned} \tag{3.10}$$

Proposition 3.3. *A foliated Lie algebroid (A, B) over $(M, \mathcal{F})$ is equivalent with a couple (E^*, W), where E^* is an $\mathcal{F}$-foliated vector subbundle of A^* and W is a homological super-vector field of degree 1 on ΠA^* such that*

i) W is foliated (projectable) with respect to the super-foliation $\bar{\mathcal{F}}$ of ΠA^* associated to the pair $(\mathcal{F}, B)$,

ii) $W|_{\Pi(A^*/E^*)} \in T\bar{\mathcal{F}} \cap T\Pi(A^*/E^*)$ and is a homological super-vector field of degree 1 on $\Pi(A^*/E^*)$, transitive over the leaves of $\mathcal{F}$.

Proof. Formula (3.6) shows that the Vaintrob super-vector field W of a foliated Lie algebroid (A,B) is a homological super-vector field (3.10) that satisfies the conditions

$$\begin{gathered}\gamma^s_{hq}=0,\ \alpha^s_{pq}=\alpha^s_{pq}(x^b),\ \alpha^a_q=\alpha^a_q(x^b),\\ \lambda^a_h=0,\ \beta^s_{hk}=0,\ rank(\beta^u_h)=dim\,\mathcal{F},\end{gathered}\tag{3.11}$$

and where the super-vector field

$$W|_{\Pi(A^*/E^*)}=\frac{1}{2}\beta^l_{hk}\bar{\eta}^h\bar{\eta}^k\frac{\partial}{\partial\bar{\eta}^l}+\beta^u_h\bar{\eta}^h\frac{\partial}{\partial y^u}\in T\bar{\mathcal{F}}\cap T\Pi(A^*/E^*)$$

is a homological super-vector field on $\Pi(A^*/E^*)$. The condition $rank(\beta^u_h)=dim\,\mathcal{F}$ is what we meant by the transitivity of W over the leaves of $\mathcal{F}$. Conversely, if we start with the data (A,E^*,W), A, which has the Lie algebroid structure of Vaintrob super-vector field W, has the subbundle $B=ann\,E^*$ and ΠA^* has the superfoliation $\bar{\mathcal{F}}$. Then, W may be represented under the form (3.10). Condition i), which means that the terms of (3.10) that contain $(\partial/\partial x^a,\partial/\partial\bar{\zeta}^q)$ depend on the coordinates $(x^a,\bar{\zeta}^q)$ only, implies the first four conditions (3.11) and condition ii) ensures the fact that B, with the Lie algebroid structure of Vaintrob super-vector field $W|_{\Pi(A^*/E^*)}$, is a foliation of A. □

4 Cohomology and deformations

The cohomology $H^*(A)$ of a Lie algebroid A is that of the differential graded algebra $(\Omega(A)=\Gamma\wedge A^*,d_A)$ of A-*forms* where, if $\lambda\in\Omega^k(A)$, then

$$\begin{aligned}d_A\lambda(s_1,...,s_{k+1})&=\sum_{j=1}^{k+1}(-1)^{j+1}\sharp_A s_j(\lambda(s_1,...,s_{j-1},s_{j+1},...,s_{k+1}))\\ &+\sum_{j<l=1}^{k+1}(-1)^{j+l}\lambda([s_j,s_l]_A,s_1,...,s_{j-1},s_{j+1},...,s_{l-1},s_{l+1},...,s_{k+1}).\end{aligned}\tag{4.1}$$

The anchor of the Lie algebroid produces a homomorphism of differential graded algebras $\sharp'_A:(\Omega(M),d)\to(\Omega(A),d_A)$ defined by

$$(\sharp'\theta)(s_1,...,s_k)=\theta(\sharp s_1,...,\sharp s_k),\quad \theta\in\Omega^k(M)$$

with the corresponding cohomology homomorphism $\sharp^*_A:H^*_{deRham}(M)\to H^*(A)$.

The general pattern of the cohomology of a foliated Lie algebroid (A,B) over a foliated base manifold $(M,\mathcal{F})$ is the same like that of a foliated manifold [23], which is why we only sketch it briefly here. Using the decomposition $A=B\oplus C$ and the bases (b_h,a_l), (b^{*h},a^{*l}) where $b_h\in\Gamma B, a_l\in\Gamma C$ and a_l are B-foliated, we get a bigrading of the A-forms such that a form of *type* (bi-degree) (s,r) (s is the C-grade and r is the B-grade) has the local expression

$$\lambda=\frac{1}{s!r!}\lambda_{l_1...l_sh_1...h_r}(x^a,y^u)a^{*l_1}\wedge...\wedge a^{*l_s}\wedge b^{*h_1}\wedge...\wedge b^{*h_r}.\tag{4.2}$$

We will denote by $\Omega^{s,r}(A)$ the space of A-forms of type (s,r).

Since B is bracket closed, one has $d_A a^{*l}(b_h, b_k) = 0$ and we see that $d_A\lambda$ has only components of the type $(s+1,r),(s,r+1),(s+2,r-1)$. Hence there exists a decomposition

$$d_A = d'_A + d''_A + \partial_A, \tag{4.3}$$

where the terms are differential operators of types $(1,0),(0,1),(2,-1)$, respectively. The property $d_A^2 = 0$ is equivalent with

$$\begin{gathered}(d''_A)^2 = 0, d'_A d''_A + d''_A d'_A = 0,\ (\partial_A)^2 = 0,\\ d'_A\partial_A + \partial_A d'_A = 0,\ (d'_A)^2 + d''_A\partial_A + \partial_A d''_A = 0.\end{gathered} \tag{4.4}$$

The cohomology of $(\Omega^{0\bullet}(A), d''_A)$ is the cohomology of the Lie algebroid B. On the other hand, an A-form λ is said to be *B-foliated* if it is of type $(s,0)$ and its local components $\lambda_{l_1,\ldots,l_s}$ are $\mathcal{F}$-foliated functions (this is an invariant condition because the bases (a_l) are B-foliated and have $\mathcal{F}$-foliated transition functions). The operator d'_A induces a cohomology of the B-foliated forms called the *basic cohomology*. Formulas (4.4) show that $(\Omega(A), d_A)$ is a semi-positive double cochain complex [23], hence, it produces a spectral sequence that connects among the previously mentioned cohomologies. Another component of the pattern is the truncated cohomology discussed in the Appendix of [26].

Example 4.1. In Example 1.2 we showed that for any Lie algebroid A the pair $(J^1A, Hom(TM,A))$ is a foliated Lie algebroid over the foliation of the basis M by points, therefore, its cohomology has the pattern described above. On the other hand, the cohomology of J^1A may be interpreted as the 1-*differentiable cohomology* of A in the sense of Lichnerowicz, who studied the 1-differentiable cohomology of many infinite dimensional Lie algebras on manifolds (e.g., see [12, 13] and the references therein; the case studied in [13] is that of the cotangent Lie algebroid of a Poisson manifold). This cohomology is defined by k-cochains $\lambda \in Hom_{\mathbb{R}}(\Gamma(\wedge^k A), C^\infty(M))$, which are differential operators of order 1 in each argument, and by the differential d defined by formula (4.1). Since a differential operator of order 1 is the composition of a $C^\infty(M)$-linear morphism by the first jet mapping j^1, the 1-differentiable cochains may be identified with usual cochains of the Lie algebroid J^1A. Furthermore, since $[j^1a_1, j^1a_2]_{J^1A} = j^1[a_1,a_2]_A$ $(a_1,a_2 \in \Gamma A)$ [5], d for 1-differentiable cochains may be identified with d_{J^1A}. Hence, $H^k_{1-diff}(A) \approx H^k(J^1A)$ as claimed. A complementary subbundle C of the foliation $Hom(TM,A)$ is given by the image of a splitting $\sigma : A \to J^1A$ of the exact sequence (1.5) and the type decomposition (4.2) corresponds to the type decomposition used by Lichnerowicz.

Like in the case of a foliation [22], we get

Proposition 4.1. (The d''_A-Poincaré Lemma) *Let (A,B) be a minimally foliated Lie algebroid. For any local A-form λ of type (u,v) $(v>0)$ such that $d''_A\lambda = 0$, one has $\lambda = d''_A\mu$ where μ is a local form of type $(u, v-1)$.*

Proof. We may proceed by induction on k, where k is the number defined by the condition that the local expression of λ does not contain the elements $a^{*k+1}, ..., a^{*q}$ ($q = rank\, A - rank\, B$) of the basis used in (4.2).

Since we are in the case of a minimally foliated algebroid, $\sharp_B : B \to T\mathcal{F}$ is an isomorphism, which induces an isomorphism between the graded differential algebra $(\Omega^{0\,\boldsymbol{\cdot}}(A), d''_A)$ and the graded differential algebra of the $\mathcal{F}$-leaf-wise forms on $(M, \mathcal{F})$. Since the d''-Poincaré lemma holds for the latter [22, 23], it holds for the former as well, which justifies the conclusion for $k = 0$.

Now, we assume that the result holds for any $k < h - 1$ ($h = 1, ..., q$) and take

$$\lambda = a^{*h} \wedge \mu + \theta$$

where μ, θ do not contain $a^{*h}, ..., a^{*q}$. Since $d''_A a^{*h} = 0$, $d''_A \lambda = 0$ if and only if $d''_A \mu = d''_A \theta = 0$. Then, the induction hypothesis yields the local expressions $\mu = d''_A \tau, \theta = d''_A \sigma$, therefore,

$$\lambda = d''_A(-a^{*h} \wedge \tau + \sigma),$$

which means that the result also holds for $k = h - 1$ and we are done. □

Obviously, the operator d''_A makes sense for V-valued A-forms, where V is any $\mathcal{F}$-foliated vector bundle on M, and the d''-Poincaré lemma (Proposition 4.1) holds for such forms as well.

The following corollary is a standard result.

Corollary 4.1. *Let (A, B) be a minimally foliated Lie algebroid and V a foliated vector bundle over $(M, \mathcal{F})$. Let Φ^k be the sheaf of germs of B-foliated, V-valued A-forms. Then one has*

$$H^l(M, \Phi^k) \approx \frac{ker(d''_A : \Omega^{k,l}(A) \otimes V \to \Omega^{k,l+1}(A) \otimes V)}{im(d''_A : \Omega^{k,l-1}(A) \otimes V \to \Omega^{k,l}(A) \otimes V)}.$$

Remark 4.1. In the terminology of [26], Proposition 4.1 is a relative Poincaré lemma. Therefore the sheaf-interpretation of the truncated cohomology given in [26] holds for minimally foliated Lie algebroids.

There exists a general construction of characteristic A-cohomology classes of a Lie algebroid [6], which mimics the Chern-Weil theory and goes as follows. Take a vector bundle $V \to M$ of rank r. An A-connection (covariant derivative) on V is an $\mathbb{R}$-bilinear operator $\nabla : \Gamma A \times \Gamma V \to \Gamma V$ that is $C^\infty(M)$-linear in the first argument and satisfies the condition

$$\nabla_a(fv) = f\nabla_a v + ((\sharp_A a)f)v, \quad (a \in \Gamma A, v \in \Gamma V, f \in C^\infty(M)).$$

The curvature operator of ∇ is

$$R_\nabla(a_1, a_2) = \nabla_{a_1}\nabla_{a_2} - \nabla_{a_2}\nabla_{a_1} - \nabla_{[a_1,a_2]_A}.$$

R_∇ is $C^\infty(M)$-bilinear and may be seen as a $V \otimes V^*$-valued A-form of degree 2. Then, $\forall \phi \in I^k(Gl(r, \mathbb{R}))$, the space of real, ad-invariant, symmetric,

k-multilinear functions on the Lie algebra $gl(r,\mathbb{R})$, the A-cohomology classes $[\phi(R_\nabla)] = [\phi(R_\nabla, ..., R_\nabla)] \in H^{2k}(A)$ are the *A-principal characteristic classes* of V. The characteristic classes are spanned by the classes $[c_k(R_\nabla)]$, where c_k is the sum of the principal minors of order k in $det(R_\nabla - \lambda Id)$, and do not depend on the choice of the connection ∇. Indeed, if ∇^0, ∇^1 are two A-connections on V, Bott's proof [1, 6] gives

$$\phi(R_{\nabla_1}) - \phi(R_{\nabla^0}) = d_A\Delta(\nabla^0, \nabla^1)\phi, \tag{4.5}$$

where

$$\Delta(\nabla^0, \nabla^1)\phi = k\int_0^1 \phi(\nabla^1 - \nabla^0, \underbrace{R_{(t\nabla^1+(1-t)\nabla^0)}, ..., R_{(t\nabla^1+(1-t)\nabla^0)}}_{(k-1)-\text{times}})dt, \tag{4.6}$$

hence the cohomology classes defined by $\phi(R^0_\nabla), \phi(R^1_\nabla)$ coincide. In particular, the use of an orthogonal A-connection (such that $\nabla g = 0$ for a Euclidean metric g on A) gives $[c_{2h-1}(R_\nabla)] = 0$. The classes $[c_{2h}(R_\nabla)]$ are the *A-Pontrjagin classes* of V. For $V = A$ one gets the principal characteristic classes of the Lie algebroid A.

In the case of a foliated Lie algebroid (A, B) over $(M, \mathcal{F})$ one can also mimic the construction of secondary characteristic classes. We define a *Bott A-connection* on a foliated vector bundle V by asking it to satisfy the condition $\nabla_b v = 0$, $\forall b \in B$, $\forall v \in \Gamma_{fol}V$. Bott A-connections always exist: take an arbitrary A-connection ∇' on V and a decomposition $A = B \oplus C$, then, define

$$\nabla_{b+c}(fv) = ((\sharp_A b)f)v + \nabla'_c(fv) \quad (f \in C^\infty(M), v \in \Gamma_{fol}V).$$

The curvature of a Bott A-connection satisfies the property

$$R_\nabla(b_1, b_2)v = 0, \forall b_1, b_2 \in B, v \in V. \tag{4.7}$$

Indeed, due to $C^\infty(M)$-linearity with respect to v, it suffices to check the property for $v \in \Gamma_{fol}V$, which is trivial.

Proposition 4.2. (The Bott Vanishing Phenomenon) *If $k > q = rank\, A - rank\, B$, $\forall \phi \in I^k(Gl(r,\mathbb{R}))$, the characteristic class defined by ϕ vanishes. Equivalently, $Pont^h_A(V) = 0$ for $h > 2q$, where $Pont^h_A(V)$ is the set of elements of total cohomological degree h generated by the A-Pontrjagin classes in the cohomology algebra $H^*(A)$.*

Proof. Property (4.7) shows that R_∇ belongs to the ideal generated by $ann\, B$, hence, if $k > q$, the A-form obtained by the evaluation of ϕ on R_∇ is zero. □

Corollary 4.2. *Let B be a Lie subalgebroid of the Lie algebroid A. If $Pont^h_A(A/B) \neq 0$ for some $h > 2(rank\, A - rank\, B)$, B is not a foliation of A.*

Now, if we take $\phi \in I^{2h-1}(Gl(r,\mathbb{R}))$ with $2h - 1 > q$, formula (4.5) yields $d_A(\Delta(\nabla^0, \nabla^1)\phi) = 0$ whenever ∇^0 is an orthogonal A-connection and ∇^1 is

a Bott A-connection on the $\mathcal{F}$-foliated vector bundle V. The A-cohomology classes $[\Delta(\nabla^0,\nabla^1)\phi]\in H^{4h-3}(A)$ are *A-secondary characteristic classes of V*. Like in [11] one can prove that these classes do not depend on the choice of ∇^0 and remain unchanged if ∇^1 is subject to a deformation via Bott connections. In the case $V=A/B$, the construction provides secondary A-characteristic classes of the foliated Lie algebroid (A,B).

Proposition 4.3. *The A-secondary characteristic classes of V are anchor images of de Rham-secondary characteristic classes.*

Proof. We may define the A-secondary characteristic classes by means of connections ∇^0,∇^1 given by $\nabla^0_a=\nabla'^0_{\sharp_A a},\nabla^1_a=\nabla'^1_{\sharp_A a}$, where ∇'_0 is a usual orthogonal connection on $V\to M$ and ∇'^1 is a Bott connection with respect to the foliated Lie algebroid $(TM,T\mathcal{F})$. Then, the conclusion holds at the level of forms. $\square$

Deformations of foliations were studied by many authors [7, 8], etc., and one of the main results is that deformations produce infinitesimal deformations, which are cohomology classes of degree 1. Here, we extend this result to deformations of a foliation B of a Lie algebroid A over $(M,\mathcal{F})$.

Definition 4.1. A *deformation* of the foliation B of A is a differentiable family of foliations $B_t\subseteq A$ $(0\leq t\leq 1)$ where $B_0=B$, each B_t is regular and $\mathcal{F}_t=\sharp_A(B_t)$ is a deformation of the foliation $\mathcal{F}=\mathcal{F}_0$.

By differentiability of B_t we mean that each $x\in M$ has a neighborhood U endowed with local cross sections $(b_h(x,t))_{h=1}^p$ of A that are differentiable with respect to (x,t) and define a basis of the local cross sections of the subbundle B_t for all $t\in[0,1]$. In particular, all the subbundles B_t have the same rank p.

The transition functions of the bases $(b_h(x,t))$ are of the form

$$\tilde{b}_h(x,t)=\beta^k_h(x,t)b_k(x,t). \tag{4.8}$$

If we apply the operator $(d/dt)_{t=0}$, we see that the correspondence

$$\eta^h b_h\mapsto [\eta^h\frac{\partial b_h(x,t)}{\partial t}\Big|_{t=0}]_{\mathrm{mod.}\,B}\quad (b_h=b_h(x,0))$$

provides a well defined $E=A/B$-valued 1-B-form Ξ. The form Ξ will be called the *infinitesimal deformation form* and it has the following invariant expression

$$\Xi(b)=[\frac{\partial\tilde{b}(x,t)}{\partial t}\Big|_{t=0}]_{\mathrm{mod.}\,B} \tag{4.9}$$

where $\tilde{b}(x,t)\in\Gamma B_t$ is an extension of $b\in B_x$; the result does not depend on the choice of the extension.

In what follows we identify the cochain complexes $(\Omega(B),d_B)$ and $(\Omega^{0,\bullet}(A),d''_A)$ by using a complementary subbundle of B in A; in particular, Ξ is also seen as a $(0,1)$-form.

Proposition 4.4. *The infinitesimal deformation form Ξ is d''_A-closed.*

Proof. We begin by deriving a new expression of Ξ. Let us fix a subbundle $C \subseteq A$ such that $A = B_t \oplus C$ holds $\forall t \in [0, \epsilon)$ (where ϵ is small enough) and consider a local basis (b_h) of B and a local basis (c_s) of C consisting of B-foliated cross sections; this yields a foliated basis $e_s = [c_s]_{\mathrm{mod.}\, B}$ of A/B. Then, we have expressions

$$b_h(x,t) = \lambda_h^k(x,t) b_k + \mu_h^s(x,t) c_s, \quad rank(\lambda_h^k) = p, \tag{4.10}$$

therefore, B_t also has a basis of the form

$$b'_h(x,t) = b_h + \theta_h^s(x,t) c_s, \quad \theta_h^s(x,0) = 0. \tag{4.11}$$

The dual cobases of the bases $(b'_h(x,t), c_s)$ are given by

$$b^{'*h}(x,t) = b^{*h}(x,0), \; c^{*s}(x,t) = c^{*s} - \theta_h^s(x,t) b^{*h} \tag{4.12}$$

(check the duality conditions

$$< b^{'*h}(x,t), b'_k(x,t) >= \delta_k^h, < c^{*s}(x,t), b'_k(x,t) >= 0,$$

$$< b^{'*h}(x,t), c_s >= 0, < c^{*s}(x,t), c_u(x,t) >= \delta_u^s).$$

From (4.11) and (4.12) we get

$$\left[\frac{\partial b'_h}{\partial t} \bigg|_{t=0} \right]_{\mathrm{mod.}\, B} = \frac{\partial \theta_h^s}{\partial t} \bigg|_{t=0} e_s$$

and

$$\Xi = \frac{\partial \theta_h^s}{\partial t} \bigg|_{t=0} b^{*h} \otimes e_s = - \frac{\partial c^{*s}(x,t)}{\partial t} \bigg|_{t=0} \otimes e_s. \tag{4.13}$$

If we denote by Ξ^C the image of Ξ by the vector bundle isomorphism $E \approx C$, which following (4.13) means that

$$\Xi^C = - \frac{\partial c^{*s}(x,t)}{\partial t} \bigg|_{t=0} \otimes c_s, \tag{4.14}$$

the conclusion of the proposition is equivalent with

$$d_A \Xi^C(b_h, b_k) = 0,$$

which we see to hold by means of the following calculation. From (4.14) and using the definition of d_A we have

$$d_A \Xi^C(b_h, b_k) = - \frac{\partial [d_A c^{*s}(x,t)(b'_h(x,t), b'_k(x,t))]}{\partial t} \bigg|_{t=0} c_s$$

$$= \frac{\partial [c^{*s}(x,t)([b'_h(x,t), b'_k(x,t)]_A)]}{\partial t} \bigg|_{t=0} c_s = \frac{\partial}{\partial t} \bigg|_{t=0} (pr_C^t [b'_h(x,t), b'_k(x,t)]_A),$$

where pr_C^t is the projection defined by the decomposition $A = B_t \oplus C$. Since B_t is closed by A-brackets, the final term of the previous calculation is equal to zero. $\square$

Continuing to follow the analogy with foliation theory, we give the following definition.

Definition 4.2. A *trivial deformation* of the foliation B of the Lie algebroid A is a deformation B_t such that $B_t = \Psi_t(B)$, where (Ψ_t, ψ_t) $(\psi_t : M \to M)$ is a family of automorphisms of A of the form $\Psi_t = exp(ta), \psi_t = exp(t\sharp_A a)$, $a \in \Gamma A$.

We recall that $\Psi_t = exp(ta)$ is a (local) 1-parameter group with respect to addition on t, which is characterized by the property [15]

$$\left.\frac{d}{dt}\right|_{t=0} (exp(ta))a' = [a, a']_A, \quad \forall a, a' \in \Gamma A. \tag{4.15}$$

Proposition 4.5. *The infinitesimal deformation form Ξ of a trivial deformation is d''_A-exact.*

Proof. For a trivial deformation we may use the local bases

$$b_h(x,t) = exp(ta)(b_h(exp(-t\sharp_A a)(x), 0),$$

which implies

$$\left.\frac{\partial b_h(x,t)}{\partial t}\right|_{t=0} = [a, b_h]_A(x).$$

Now, using a decomposition $a = \alpha^h b_h + \beta^s c_s$, where (b_h, c_s) is the basis used in (4.10), the original definition of Ξ shows that the corresponding form Ξ^C is given by

$$\Xi^C(\eta^h b_h) = pr_C(\eta^h [a, b_h]_A) = -\eta^h(\sharp_A b_h(\beta^s))c_s = -d''_A(pr_C a), \tag{4.16}$$

which is equivalent with the required result. $\square$

Definition 4.3. Let (A, B) be a foliated Lie algebroid. Two deformations $B_t, \bar{B}_t$ of B are *equivalent* if there exists a cross section $a \in \Gamma A$ such that $\bar{B}_t = exp(ta)(B_t)$, i.e., if they can be deduced from one another by composition with a trivial deformation.

Now, we get the following result:

Proposition 4.6. *Two equivalent deformations have d''_A-cohomologous infinitesimal deformation forms.*

Proof. For two equivalent deformations $B_t, \bar{B}_t$ there are local bases related as follows

$$\bar{b}_h(x,t) = exp(ta)(b_h(x,t)).$$

By applying the operator $(\partial/\partial t)_{t=0}$ to this relation and using the computation (4.16) we get the following relation between the corresponding infinitesimal deformation forms

$$\bar{\Xi}^C = \Xi^C - d''_A(pr_C a).$$

$\square$

The cohomology class $[\Xi] \in H^1(B; A/B)$ (the first cohomology space of A/B-valued B-forms) is the *infinitesimal deformation up to equivalence* of the foliation B of the Lie algebroid A.

5 Integration of foliated Lie algebroids

In this section we briefly discuss the integrability of a foliated Lie algebroid.

Definition 5.1. Let $l, r : G \rightrightarrows M$ be a Lie groupoid (l, r are the target (left) and source (right) projections, respectively), where the unit manifold M is endowed with a foliation $\mathcal{F}$. We call G an *$\mathcal{F}$-foliated Lie groupoid* if it is endowed with a foliation $\mathcal{G}$ that has the following properties:
(i) the leaves of $\mathcal{G}$ are contained in the left fibers,
(ii) $\mathcal{G}$ is invariant by left translations,
(iii) the right projection $r : (G, \mathcal{G}) \to (M, \mathcal{F})$ is a foliated, leaf-wise submersion.

It is easy to get the following integrability result.

Proposition 5.1. *Let $l, r : (G, \mathcal{G}) \rightrightarrows (M, \mathcal{F})$ be a foliated Lie groupoid. Then, the Lie algebroid $A(G)$ has a natural structure of a foliated Lie algebroid. Conversely, if (A, B) is a foliated Lie algebroid and A is integrable by the left-fiber connected Lie groupoid G, then G has a natural foliation $\mathcal{G}$ that makes G into a foliated Lie groupoid such that the restriction to the leaves of $\mathcal{G}$ of the construction of $A(G)$ produces a Lie subalgebroid that is isomorphic to B.*

Proof. If we start with the foliated groupoid $(G, \mathcal{G})$ and construct the Lie algebroid $A(G)$ by means of the left-invariant vector fields, the invariant vector fields that are tangent to the leaves of $\mathcal{G}$ produce a Lie subalgebroid $B \subseteq A(G)$.

Furthermore, if we denote by T^lG the tangent bundle to the l-fibers we get the exact sequence

$$0 \to T\mathcal{G} \to T^lG \to T^lG/T\mathcal{G} \to 0,$$

where the quotient is a $\mathcal{G}$-foliated bundle. The foliated cross sections of T^lG are the infinitesimal automorphisms of $T\mathcal{G}$ and (like for any foliation) span the bundle T^lG. If the previous sequence is quotientized by left translations we get the exact sequence

$$0 \to B \to A(G) \to A(G)/B \to 0$$

and we see that the pair $(A(G), B)$ satisfies the foliated generation condition. Then, the anchor of $A(G)$ is induced by the differential r_* of the right projection and its restriction to B is onto $T\mathcal{F}$ because of condition (iii) of Definition 5.1. Therefore, $(A(G), B)$ is a foliated Lie algebroid over $(M, \mathcal{F})$.

Conversely, if we start with a foliated Lie algebroid (A, B) over $(M, \mathcal{F})$ and $A = A(G)$ for the Lie groupoid G, we get the foliation $\mathcal{G}$ by asking it to be tangent to the left invariant vector fields that produce cross sections of B (see Lemma 2.1 of [17]). Then, the conditions required in Definition 5.1 obviously

hold. In particular, condition (iii) follows from the foliated-generation condition of (A, B) together with the fact that $im\,\sharp_B = T\mathcal{F}$. $\square$

6 Foliated Courant algebroids

We do not intend to develop a theory of foliated Courant algebroid, but, we would like to clarify how such a notion should be defined. We refer to [14] for the general theory of Courant algebroids.

Definition 1.1 of a transversal-Lie algebroid suggests the following natural, analogous definition.

Definition 6.1. A *transversal-Courant algebroid* over the foliated manifold $(M, \mathcal{F})$ is a foliated vector bundle $E \to M$ endowed with a symmetric, non degenerate, foliated, inner product $g \in \Gamma_{fol} \odot^2 E^*$, with a foliated morphism $\sharp_E : E \to \nu\mathcal{F}$ called the anchor and a skew-symmetric bracket $[\,,\,]_E : \Gamma_{fol}E \times \Gamma_{fol}E \to \Gamma_{fol}E$, such that the following conditions are satisfied:

1) $\sharp_E[e_1, e_2]_E = [\sharp_E e_1, \sharp_E e_2]_{\nu\mathcal{F}}$,
2) $im(\sharp_g \circ^t \sharp_E) \subseteq ker\,\sharp_E$,
3) $\sum_{Cycl}[[e_1, e_2]_E, e_3]_E = (1/3)\partial \sum_{Cycl} g([e_1, e_2]_E, e_3)$, $\partial f = (1/2)\sharp_g({}^t\sharp_E df)$,
4) $[e_1, fe_2]_E = f[e_1, e_2]_E + ((\sharp_E e_1)f)e_2 - g(e_1, e_2)\partial f$,
5) $(\sharp_E e)(g(e_1, e_2)) = g([e, e_1]_E + \partial g(e, e_1), e_2) + g(e_1, [e, e_2]_E + \partial g(e, e_2))$.

In these conditions, $e, e_1, e_2, e_3 \in \Gamma_{fol}E$, $f \in C^\infty_{fol}(M, \mathcal{F})$ and t denotes transposition.

Using local, foliated, g-canonical bases, it is easy to see that ${}^t\sharp_E(\lambda) \in \Gamma_{fol}E^*$, $\forall\lambda \in \Gamma_{fol}(ann\,T\mathcal{F})$. In particular, $\forall f \in C^\infty_{fol}(M, \mathcal{F})$, $\partial f \in \Gamma_{fol}E$.

Like for the Courant algebroids, which are obtained if $\mathcal{F}$ is the foliation of M by points, we will say that the transversal-Courant algebroid E is *transitive* if $\sharp_E$ is surjective and it is *exact* if it is transitive and $rank\,E = 2rank\,\nu\mathcal{F}$. The reason for this name is that E is exact if and only if the sequence of vector bundles

$$0 \longrightarrow ann\,T\mathcal{F} \overset{\sharp_{(E,g)}}{\longrightarrow} E \overset{\sharp_E}{\longrightarrow} \nu\mathcal{F} \longrightarrow 0, \tag{6.1}$$

where $\sharp_{(E,g)} = \sharp_g \circ^t \sharp_E$ (therefore, $\partial f = \sharp_{(E,g)}df$) is exact. (This follows like for Courant algebroids, e.g., Section 3 of [24].)

Remark 6.1. For an exact Courant algebroid any choice of a splitting of the exact sequence (6.1) that has an isotropic image yields an isomorphism $E \approx (TM \oplus T^*M)$ where the latter is endowed with a twisted Courant bracket [2]. The brackets of the transitive Courant algebroids were determined in [19, 24] and may be made explicit by using a splitting as above. These results extend to transversal Courant algebroids if we add the hypothesis that the exact sequence (6.1) has a foliated splitting, which is no more an automatic fact.

Remark 6.2. If E is a transversal-Courant algebroid over $(M, \mathcal{F})$, the local projected bundles $\{E_{Q_U}\}$ are Courant algebroids over the local transversal manifolds Q_U of $\mathcal{F}$. In the case of a submersion $M \to Q$ with connected and simply connected fibers, $E \to M$ projects to a Courant algebroid over Q.

Example 6.1. For any foliated manifold $(M, \mathcal{F})$ the vector bundle $\nu\mathcal{F} \oplus \nu^*\mathcal{F} = \nu\mathcal{F} \oplus ann\, T\mathcal{F}$ has the structure of a transversal-Courant algebroid induced by the classical Courant structure of the local transversal manifolds Q_U of Remark 6.2.

Example 6.2. Let A be a Courant algebroid over the foliated manifold $(M, \mathcal{F})$ and let B be a subbundle of A that is a Courant algebroid with respect to the induced Courant bracket, anchor and metric. In particular, the metric induced in B must be non degenerate and $A = B \oplus C$, $C = B^{\perp_g}$. A cross section $c \in \Gamma C$ will be called B-foliated if, $\forall b \in \Gamma B$, $[b, c]_A \in \Gamma B$. By the axioms of Courant algebroids (see condition 4) of Definition 6.1) we see that, on one hand, if the previous condition holds for b it also holds for fb ($f \in C^\infty(M)$ and, on the other hand, if c is B-foliated and $f \in C^\infty_{fol}(M, \mathcal{F})$, fc is foliated (use $g(b, c) = 0$). (The definition of a B-foliated cross section is not correct for an arbitrary $a \in \Gamma A$.) Now, it makes sense to assume that the pair (A, B) satisfies the following conditions (analogous to those in Definition 1.2): i) $\sharp_A(B) = T\mathcal{F}$, ii) ΓC is locally spanned by the B-foliated cross sections. Like for Lie algebroids (Section 1), these conditions show that C has a natural structure of a foliated vector bundle and we may add condition iii) $g|_C \in \Gamma_{fol} \odot^2 C^*$. From i), we get ${}^t\sharp_A(ann\, T\mathcal{F}) \subseteq ann\, B = C^*$, whence, $\partial f \in \Gamma C$, $\forall f \in C^\infty_{fol}(M, \mathcal{F})$. Furthermore, since A is a Courant algebroid we have (see condition 5), Definition 6.1)

$$(6.2)\quad (\sharp_A c_1)(g(b, c_2)) = 0 = g([c_1, b]_A + \partial g(c_1, b), c_2) + g(b, [c_1, c_2]_A + \partial g(c_1, c_2)),$$

where $b \in \Gamma B, c_1, c_2 \in \Gamma C$ and c_1, c_2 are B-foliated. In view of condition iii), (6.2) becomes $g(b, [c_1, c_2]_A) = 0$, therefore, $[c_1, c_2]_A \in \Gamma C$. Moreover, using axiom 3) for the Courant algebroid A and for the triple (c_1, c_2, b) instead of (e_1, e_2, e_3) we see that the bracket $[c_1, c_2]_A$ is B-foliated. Thus, conditions i), ii), iii) imply that $(C, \sharp_A|_C, [\,,\,]_A|_{\Gamma_{fol} C})$ is a transversal-Courant algebroid over $(M, \mathcal{F})$.

We have formulated the above example because, at the first sight, it could indicate the way to the notion of a foliated Courant algebroid. However, it seems that there are no corresponding, interesting, concrete examples and we shall propose another procedure below.

Assume that B is a g-isotropic subbundle of the Courant algebroid A of basis $(M, \mathcal{F})$ such that ΓB is closed by A-brackets and consider the g-coisotropic subbundle $C = B^{\perp_g} \supseteq B$. By replacing c_1 by $b' \in \Gamma B$ in (6.2), we see that ΓB is closed by A-brackets if and only if

$$(6.3)\qquad [b, c]_A \in \Gamma C, \qquad \forall b \in \Gamma B, \forall c \in \Gamma C.$$

Like in Example 6.2, a cross section $c \in \Gamma C$ will be called B-foliated if, $\forall b \in \Gamma B$, $[b, c]_A \in \Gamma B$ and this definition is correct. We will denote by $\Gamma_B C$ the space of B-foliated cross sections of C.

Definition 6.2. Let A be a Courant algebroid over the foliated manifold $(M, \mathcal{F})$. A subbundle $B \subseteq A$ will be called a *foliation* of A if the following conditions are satisfied:

i) B is a g-isotropic and ΓB is closed by A-brackets,

ii) $\sharp_A(B) = T\mathcal{F}$,

iii) ΓC, $C = B^{\perp_g}$, is locally spanned by the B-foliated cross sections,

iv) for any pair of B-foliated cross sections $c_1, c_2 \in \Gamma C$ one has $g(c_1, c_2) \in C^\infty_{fol}(M, \mathcal{F})$.

A *foliated Courant algebroid* is a pair (A, B) where A is a Courant algebroid and B is a foliation of A.

Proposition 6.1. *If (A, B) is a foliated Courant algebroid over $(M, \mathcal{F})$, the vector bundle $E = C/B$, $C = B^{\perp_g}$, inherits a natural structure of an $\mathcal{F}$-transversal-Courant algebroid.*

Proof. Since B is isotropic, there exist decompositions

$$A = B \oplus S \oplus B',\ B \oplus S = C,\ (B \oplus B') \perp_g S, \tag{6.4}$$

where B' is isotropic and g is non degenerate on the components $B \oplus B'$ and S. We fix such a decomposition and define a structure of transversal-Courant algebroid on S.

Take

$$g_S = g|_S,\ \sharp_S = \sharp_A\,(mod.\, T\mathcal{F}),\ [\,,\,]_S = pr_S[\,,\,]_A,$$

where the projection on S is defined by (6.4). Like in the proof of Proposition 1.2, properties ii) and iii) of Definition 6.2 yield a foliated structure of S such that $s \in \Gamma S$ is foliated with respect to this structure if and only if s is B-foliated. We check that, $\forall s \in \Gamma_B C$, $\sharp_S s$ is $\mathcal{F}$-foliated. Indeed, if $Y \in \Gamma T\mathcal{F}$, property ii) allows us to write $Y = \sharp_A b$, $b \in \Gamma B$ and we have

$$[Y, \sharp_A s]_A = [\sharp_A b, \sharp_A s]_A = \sharp_A [b, s]_A \in \Gamma T\mathcal{F},$$

therefore $\sharp_A s$ is a foliated vector field on $(M, \mathcal{F})$. Thus, $\sharp_S$ is a foliated morphism. Furthermore, property iv) implies that g_S is a foliated metric.

The decomposition (6.4) also gives $A^* = B^* \oplus S^* \oplus B^{'*}$ and we have

$$\flat_g(B) = B^{'*},\ \flat_g(S) = S^*,\ \flat_g(B') = B^*.$$

If $\gamma \in ann\, T\mathcal{F}$, ${}^t\sharp_A \gamma$ vanishes on B, because of property ii), hence, ${}^t\sharp_A \gamma \in S^* \oplus B^{'*}$ and $\sharp_g^t \sharp_A \gamma \in C$. In particular, $\forall f \in C^\infty_{fol}(M, \mathcal{F}, \partial f \in \Gamma C$. This fact allows us to use again (6.2), while taking $s_1, s_2 \in \Gamma_B C$ instead of c_1, c_2, and we see that $[s_1, s_2]_A \in \Gamma C$. Now, if we write down the equality iii), Definition 6.1 for the Courant algebroid A and for e_1, e_2, e_3 replaced by $s_1, s_2 \in \Gamma_B C, b \in \Gamma B$, the right hand side vanishes and we remain with $[[s_1, s_2]_A, b]_A = 0$, which shows that $[\,,\,]_S$ takes foliated cross sections to foliated cross sections.

Thus, S is endowed with all the structures required by Definition 6.1 and it remains to check the conditions 1)-5). It is easy to check that $\partial_S = \partial_A$

on $C^\infty_{fol}(M,\mathcal{F})$. Accordingly, 1)-5) are implied by the similar properties of the Courant algebroid A (in particular, for 2) we may use the fact that this condition is equivalent with $g(\partial_S f_1, \partial_S f_2) = 0, \forall f_1, f_2 \in C^\infty_{fol}(M,\mathcal{F})$ [14]). The conclusion of the proposition follows by transferring the structure obtained on S to E via the natural isomorphism $E \approx S$. □

Corollary 6.1. *If (A, B) is a foliated Courant algebroid over $(M,\mathcal{F})$ and if $\mathcal{F}$ consists of the fibers of a submersion $M \to Q$ with connected and simply connected fibers, the vector bundle $E = B^{\perp_g}/B$ projects to a Courant algebroid on Q.*

Remark 6.3. If B is a subbundle of the Courant algebroid A that satisfies conditions i), ii) of Definition 6.2 and if we ask the quotient bundle A/B to be a transversal-Courant algebroid such that its metric, anchor and bracket be induced by those of A, then, obviously, B is a foliation of A.

The previous proposition and corollary show the interest of the notion of a foliated Courant algebroid. The following examples show that Definition 6.2 is reasonable. In these examples $T^{big}M = TM \oplus T^*M$ is the Courant algebroid defined in [4]; the anchor is the projection on TM and the metric and bracket are defined by

$$g((X_1,\alpha_1),(X_2,\alpha_2)) = \frac{1}{2}(\alpha_1(X_2) + \alpha_2(X_1)), \tag{6.5}$$

$$[(X_1,\alpha_1),(X_2,\alpha_2)] = ([X_1,X_2], L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + \frac{1}{2}d(\alpha_1(X_2) - \alpha_2(X_1))). \tag{6.6}$$

Example 6.3. A regular Dirac structure $D \subseteq T^{big}M$ is a foliation of the Courant algebroid $T^{big}M$ over $(M,\mathcal{E})$, where $\mathcal{E}$ is the characteristic foliation of D. In this case the quotient bundle is $D^{\perp_g}/D = 0$.

Example 6.4. If $\mathcal{F}$ is a foliation of M, $T\mathcal{F}$ is a foliation of the Courant algebroid $T^{big}M$. Indeed, using (6.5) we get $C = T^{\perp_g}\mathcal{F} = TM \oplus ann\, T\mathcal{F}$. A pair $(X,\gamma) \in \Gamma C$ is $T\mathcal{F}$-foliated if and only if X is an $\mathcal{F}$-foliated vector field and γ is an $\mathcal{F}$-foliated 1-form. The conditions of Definition 6.2 are obviously satisfied and the corresponding transversal-Courant algebroid is $E = C/T\mathcal{F} = \nu\mathcal{F} \oplus ann\, T\mathcal{F}$, already mentioned in Example 6.1. Notice that $T\mathcal{F}$ remains a foliation of $T^{big}M$ if the Courant bracket is twisted by means of a closed 3-form Φ such that $i(Y)\Phi = 0, \forall Y \in T\mathcal{F}$ [20].

Example 6.5. Let $\mathcal{F}$ be a foliation of M and $\theta \in \Omega^2(M)$ be a closed 2-form. From [25], it is known that $E_\theta = \{(Y, i(Y)\theta) \,/\, Y \in T\mathcal{F}\}$ is an isotropic subbundle of $T^{big}M$, which is closed by the Courant bracket (6.6), and its orthogonal bracket is $E'_\theta = \{(X, i(X)\theta + \gamma) \,/\, X \in TM, \gamma \in ann\, T\mathcal{F}\}$. A simple computation gives

$$[(Y, i(Y)\theta), (X, i(X)\theta + \gamma)] = ([Y,X], i([Y,X])\theta + L_Y\gamma).$$

Accordingly, $(X, i(X)\theta+\gamma) \in \Gamma E'_\theta$ is E_θ-foliated if and only if $X \in \Gamma_{fol}TM, \gamma \in \Omega^1_{fol}(M)$ and, like in Example 6.4, the conditions of Definition 6.2 are satisfied and E_θ is a foliation of $T^{big}M$. Again, the corresponding quotient bundle is $E'_\theta/E_\theta \approx \nu\mathcal{F} \oplus ann\, T\mathcal{F}$.

Below, we prove a result that has the flavor of a reduction and was inspired by [27]. Let B be a foliation of the Courant algebroid A over the foliated manifold $(M, \mathcal{F})$. Let N be a submanifold of the base manifold M, and consider the restricted vector bundles $A_N, B_N, C_N = B_N^{\perp_{g_A}}$.

Proposition 6.2. *With the notation above, assume that:*

(i) N is transversal to and has a clean intersection with the leaves of the foliation $\mathcal{F}$,

(ii) (the reduction hypothesis) $\sharp_A(C_N) \subseteq TN$.

Then, the vector bundle $E_N = C_N/B_N$ is a transversal-Courant algebroid over $(N, \mathcal{F}' = \mathcal{F} \cap N)$.

Proof. Since B is a foliation of A, there exist local bases $(b_h \in \Gamma B, a_u \in \Gamma_B C)$ of ΓC and the induced local bases of cross sections of $E = C/B$ have local transition functions of the form $[\tilde{a}_u]_{\mathrm{mod.}\,B} = \alpha_u^v [a_u]_{\mathrm{mod.}\,B}$ such that α_u^v are constant on the leaves of $\mathcal{F}$. Then, α_u^v are constant on the leaves of $\mathcal{F}'$ too and the local bases $[a_u|_N]_{\mathrm{mod.}\,B_N}$ of ΓE_N define an $\mathcal{F}'$-foliated structure on E_N. Using these bases, we see that a cross section of E_N is foliated with respect to the $\mathcal{F}'$-foliated structure of E_N if and only if, locally, it is of the form $[c|_N]_{\mathrm{mod.}\,B_N}$ where c is B-foliated. Moreover, the metric induced on E_N by the metric g of A obviously is a foliated metric.

In view of the reduction hypothesis, the mapping

$$[c]_{\mathrm{mod.}\,B} \mapsto [\sharp_A c]_{\mathrm{mod.}\,T\mathcal{F}} \quad (c \in \Gamma C)$$

produces a vector bundle morphism $\sharp_{E_N} : E_N \to \nu\mathcal{F}'$, which will be the anchor of the required Courant structure on E_N. Like in the proof of Proposition 6.1, we can see that this anchor is a foliated morphism.

Now, we will show the existence of a bracket $[\,,\,]_{C_N} : \Gamma C_N \times \Gamma C_N \to \Gamma E_N$ induced by the Courant bracket of A. The definition of this bracket is

$$[c_1, c_2]_{C_N}(x) = [\tilde{c}_1, \tilde{c}_2]_A(x)\ (mod.\, B_x), \quad x \in N, \tag{6.7}$$

where $\tilde{c}_1, \tilde{c}_2$ are arbitrary extensions of c_1, c_2 to ΓC. In order to prove that the result does not depend on the choice of the extensions it suffices to show that if $\tilde{c}_2|_N = 0$ then $[\tilde{c}_1, \tilde{c}_2]_A(x) \in B_x$. To see that, take a local basis $(b_h \in \Gamma B, a_u \in \Gamma_B C)$ of ΓC and put $\tilde{c}_2 = f^h b_h + k^u a_u$ where f^h, k^u vanish on N. Using the axioms of a Courant algebroid (see 4) of Definition 6.1), under the conditions above, we remain with

$$[\tilde{c}_1, \tilde{c}_2]_A(x) = -\sum_u g_x(\tilde{c}_1(x), a_u(x))\partial k^u(x).$$

Furthermore, for any $c \in \Gamma C$ we have

$$g(c, \partial k^u) = \frac{1}{2}(\sharp_A c)k^u = 0$$

(because $\sharp_A c \in TN$ and $k^u|_N = 0$), therefore, $\partial k^u(x) \in B_x$, which ends the proof of the correctness of the definition of the bracket (6.7).

The bracket (6.7) produces a bracket of foliated cross sections of E_N as follows. If $\gamma \in \Gamma_{fol}E_N$, then, in a neighborhood of $x \in N$, we may write $\gamma = \sum_u \varphi^u(x^a)[a_u|_N]_{\mathrm{mod.}\,B_N}$, which implies that γ may be represented by a cross sections of C_N that has the B-foliated, local extension $\tilde{\gamma} = \sum_u \varphi^u(x^a)a_u$. If we put

$$[\gamma_1, \gamma_2]_{E_N}(x) = [\tilde{\gamma_1}|_N, \tilde{\gamma_2}|_N]_{C_N}(x) = [\tilde{\gamma_1}, \tilde{\gamma_2}]_A(x)\ (mod.\, B_x), \tag{6.8}$$

we get a well defined bracket on $\Gamma_{fol}E_N$.

Thus, on E_N we have all the components required by the definition of a transversal-Courant algebroid over $(N, \mathcal{F}')$ and, moreover, these components may be seen as restrictions to N of the components of the transversal-Courant algebroid E over $(M, \mathcal{F})$ given by Proposition 6.1. Hence, condition 1)-5) of Definition 6.1 are satisfied and we are done. □

We finish by the observation that a definition similar to Definition 6.1 may be given for a notion of *holomorphic Courant algebroid*; we just have to replace the word "foliated" by "holomorphic" everywhere. Furthermore, if we state Definition 6.2 for complex vector bundles A, B over a complex analytic manifold M, and with $T\mathcal{F}$ replaced by the anti-holomorphic tangent bundle of M, we get the notion of a foliation of a complex Courant algebroid. The corresponding quotient E will be a holomorphic Courant algebroid. Examples 6.4, 6.5 can be adapted to the holomorphic case

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